UNIVERSITY OF NAIROBI
COLLEGE OF BIOLOGICAL AND PHYSICAL SCIENCES

SCHOOL OF MATHEMATICS LECTURE NOTES

SMA 103: CALCULUS I

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Figure 1: Photo of Author in Syracuse, NY, USA 2004.

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About the Author

B.M. Nzimbi is a Senior Lecturer in Pure Mathematics at the University of Nairobi. Dr. Nzimbi received his B.sc in Mathematics and Computer Science from the University of Nairobi (1995), Msc(Pure Mathematics) from the University of Nairobi (1999), Msc(Mathematics) from Syracuse University (New York, USA) (2004), and his PhD in Pure Mathematics from the University of Nairobi (2009), where he wrote his research thesis in the area of Operator Theory under the direction of Prof. J.M. Khalagai. Before joining the University of Nairobi, he held a position at Catholic University of Eastern Africa (CUEA), where he was a part-time lecturer.

Dr. Nzimbi has authored, co-authored and published numerous articles/research papers in professional and refereed journals in the areas of Operator Theory and Differential Geometry. He is the author of the textbooks "Linear Algebra I" and "Linear Algebra II", which are extensively used in the Outreach and Distance Learning(ODL) programme at the University of Nairobi.

Preface

This volume aims to present calculus in an intuitive yet intellectually satisfying way and to illustrate the many applications of calculus to the pure sciences and management sciences. The only co-requisite for mastering the material in the book are SMA 101: Basic Mathematics and an interest in mathematics and a willingness occasionally to suspend disbelief when a familiar idea occurs in an unfamiliar guise. But only an exceptional student would profit from reading the book unless he/she has previously acquired a fair working knowledge of elementary set theory, algebra and geometry. This book is a development of various courses designed for first year students of science at the University of Nairobi, whose preparation has been some rudimentary knowledge of set theory, algebra and geometry.

What is Calculus?

Algebra and geometry are useful tools for describing relationships between static quantities. However, they do not involve concepts appropriate for describing how quantity *changes*. For this we need new mathematical operations that transcend or go beyond algebra and geometry. We require operations that measure the way related quantities change.
Calculus provides the tools for describing motion quantitatively. It introduces two new operations called *differentiation* and *integration* which are inverses of each other: what differentiation does, integration undoes. The process of differentiation is closely tied to the geometric problem of finding tangent lines. Integration is related to the geometric problem of finding areas of regions with curved boundaries. These two concepts defined in terms of the concept of a *limit*. This will be developed in Chapter 1, and marks the beginning of calculus.

Origins of Calculus

Calculus was invented independently by two 17th-century mathematicians: Isaac Newton and Gottfried Wilhelm Leibniz.

Objectives

At the end of this course unit the learner will be able to:

- Understand a the concept of a limit of a function and how to compute the same.
- Appreciate the concept of a continuous function.
- Appreciate and apply the concepts of derivatives and antiderivatives.

Acknowledgements

This book could not have been written without direct and indirect assistance from many sources. I had the good fortune of interacting with a number of special people and institutions. I seize this opportunity to express my gratitude to all of them. It is a pleasure to acknowledge with great pleasure the encouragement I received for this project from my former students and colleagues at the University of Nairobi and beyond. I wish to acknowledge, with thanks, my colleagues who have encouraged me to acquire expert knowledge of the typesetting software LATEX. As always, I am grateful to the University of Nairobi, whose facilities have made it possible to write this book. They have also provided an environment that helped me develop academically and professionally and given me the resources needed to make this book a success. I am also thankful to the School of Mathematics for giving me the opportunity to teach calculus for years, which has indeed helped this project to take shape. I would also like to thank the Open and Distance Learning(ODL) Programme for giving me the opportunity at several review sessions at Multimedia University, Mbagathi and Kenya Wildlife Training Institute, Naivasha to organize and typeset this book. In these circumstances it would have been extraordinarily slothful not to produce a book. I read the galleys myself and checked the calculations in the text. If the book contains any errors or stylistic misjudgement or shortcomings, they are entirely mine! Last, but not least, I am grateful to my family for their continuing and unwavering support. To them I dedicate this book.

Dr. Bernard Mutuku Nzimbi

Nairobi, September 2014

Chapter 1

PRELIMINARIES FOR CALCULUS

> *Neglect of mathematics works injury to all knowledge, since he who is ignorant of it cannot know the other sciences or the things of this world. And what is worse, men who are thus ignorant are unable to perceive their own ignorance and so do not seek a remedy.* —**Roger Bacon**

Functions are basic to the study of mathematics and many other sciences and are found in everyday life situations.
To address functions adequately we need to lay a foundation. We start by introducing the notion of sets.

1.1 Elementary Set Theory

Definition 1.1.1. A **set** is any well-defined *collection*, *aggregate*, *class* or *conglomerate* of objects.

These objects (which may be cities, years, numbers, letters of the alphabet, or anything else) are called *elements* of the set, and are often said to be *members* of the set.
A set is often specified by
⊙ listing its elements inside a pair of braces or curly brackets or parentheses.
⊙ means of a property of its elements.

Example 1.1.2. The set whose elements are the first six letters of the alphabet is written

$$\{a, b, c, d, e, f\}$$

Example 1.1.3. The set whose elements are the even integers between 1 and 11 is written

$$\{2, 4, 6, 8, 10\}$$

We can also specify a set by giving a description of its elements (without actually listing the elements).

Example 1.1.4. The set $\{a, b, c, d, e, f\}$ can also be written

$$\{The\ first\ six\ letters\ of\ the\ alphabet\}$$

1.1.1 Notation and Terminology

For convenience, we usually denote sets by capital letters of the alphabet A, B, C and so on. We use lowercase letters of the alphabet to represent elements of a set. For a set A, we write $x \in A$ if x is a member of A or belongs to A. We write $x \notin A$ to mean that x is not a member of A or does not belong to A.

Example 1.1.5. If $\mathbb{E}$ denotes the set of even integers, then $4 \in \mathbb{E}$ but $7 \notin \mathbb{E}$.

Definition 1.1.6. An **empty set** is a set with no elements.

An empty set is usually denoted by $\emptyset$. It is a set that arises in a variety of disguises.

Definition 1.1.7. Let A and B be two sets. If every element of A is an element of B, we say that A is a **subset** of B, and we write $A \subseteq B$. We also say that A is contained in B.

Definition 1.1.8. If $A \subseteq B$ and $B \subseteq A$, then we say that A and B are **equal**, and write $A = B$.

Definition 1.1.9. If $A \subseteq B$ and $A \neq B$, then we say that A is a **proper subset** of B are equal, and write $A = B$, or A is **properly contained** in B, and write $A \subset B$.

Definition 1.1.10. (**Cardinality of a set**). The number of elements in a set A is called the **cardinality** of A, and is denoted $n(A)$ or $|A|$.

Definition 1.1.11. (**Universal set**). A **universal set** denoted by $\mathcal{U}$ is a set which contains all elements under consideration. That is, it contains all other sets under consideration. It is also called the **universe of discourse** or simply **universe**.

1. **Complement of a set**

Let $\mathcal{U}$ be the universal set and A be any set. The **complement** of A, written A^{C} is defined as

$$A^{\mathsf{C}} = \Big\{x \in \mathcal{U} : x \notin A\Big\}$$

Example 1.1.12. Let the universal set be $\mathcal{U} = \{0, 1, 2, 3, 5, 6\}$ and $A = \{3, 5\}$. Then $A^{\mathsf{C}} = \{0, 1, 2, 6\}$.

2. **Union of sets**

Let A and B be sets. The **union** of A and B, denoted by $A \cup B$ is

$$A \cup B = \Big\{x : x \in A \text{ } or \text{ } x \in B \text{ } or \text{ } both\Big\}$$

Example 1.1.13. If $A = \{3, 5, 7\}$ and $B = \{x, y, t\}$, them $A \cup B = \{3, 5, 7, x, y, t\}$.

3. **Intersection of sets**

Let A and B be sets. The **intersection** of A and B, denoted by $A \cap B$ is

$$A \cap B = \Big\{x : x \in A \text{ } and \text{ } x \in B\Big\}$$

Example 1.1.14. If $A = \{1, 3, Tom, Mary\}$ and $B = \{3, x, y, t, Mary\}$, them $A \cap B = \{3, Mary\}$.

4. **Set Difference**
Let A and B be sets. The **set difference** or relative complement of A with respect to B, denoted by $A - B$ is defined as

$$A - B = \Big\{x : x \in A \text{ } and \text{ } x \notin B\Big\}$$

Example 1.1.15. If $A = \{New\ York, Cairo, Mumbai, Seoul, Beijing, Moscow, London\}$ and $B = \{Nairobi, Kigali, Pretoria, beijing, Harare, Paris, London\}$. Then

$$A - B = \{New\ York, Cairo, Mumbai, Seoul, Moscow\}$$

and

$$B - A = \{Nairobi, Kigali, Pretoria, Harare, Paris\}$$

5. **Cartesian Product of Sets**
Let A and B be sets. The **Cartesian product** of A and B, denoted by $A \times B$ is defined as

$$A \times B = \Big\{(a, b) : a \in A \text{ } and \text{ } b \in B\Big\}$$

Example 1.1.16. If $A = \{0, 1, 2\}$ and $B = \{a, b\}$, them

$$A \times B = \Big\{(0, a), (0, b), (1, a), (1, b), (2, a), (2, b)\Big\}$$

$$A \times A = \Big\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)\Big\}$$

Example 1.1.17. Let $\mathbb{R}$ be the set of real numbers. Then the Cartesian product of

$$\mathbb{R} \times \mathbb{R} = \Big\{(x, y) : x, y \in \mathbb{R}\Big\}$$

This is also denoted as $\mathbb{R}^2$ and read as "`R two`". This is the two-dimensional Cartesian plane or simply the xy-plane.
The Cartesian product

$$\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \Big\{(x, y, z) : x, y, z \in \mathbb{R}\Big\}$$

This is also denoted as $\mathbb{R}^3$ and read as "`R three`", and is the three-dimensional Euclidean space.

1.1.2 Some Special Number Sets

There are certain sets of numbers that appear frequently in mathematics and in other sciences.
1. **The Natural Numbers,** $\mathbb{N}$

$$\mathbb{N} = \{1, 2, 3, ...\}$$

2. **The Whole Numbers**, $\mathbb{W}$

$$\mathbb{W} = \{0, 1, 2, 3, ...\}$$

3. **The set of Integers**, $\mathbb{Z}$

$$\mathbb{Z} = \{..., -4, -3, -2, -1, 0, 1, 2, 3, 4, ...\}$$

4. **The Rational Numbers**, $\mathbb{Q}$indexRational!numbers

$$\mathbb{Q} = \{\frac{a}{b} : a, b \in Z, b \neq 0, gcd(a, b) = 1\}$$

1.1.3 Fundamental Operations on Sets

5. **The Irrational Numbers**, $\mathbb{Q}^{\complement}$indexIrrational!numbers

$$\mathbb{Q}^{\complement} = \{t : t \notin \mathbb{Q}\}$$

Note that $\mathbb{Q} \cap \mathbb{Q}^{\complement} = \emptyset$.
6. **The Real Numbers**, $\mathbb{R}$
indexReal!numbers A real number is a number which is either rational or irrational. That is $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^{\complement}$. The set of real numbers is also called the **real number line**.
7. **Intervals in** $\mathbb{R}$. Let $a, b, x \in \mathbb{R}$. Then
$(a, b) = \{x \in \mathbb{R} : a < x < b\}$ is called the *open interval* between a and b.
$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$ and $(a, b] = \{x \in \mathbb{R} : ax \leq b\}$ are called the *half-open* or *half-closed intervals* between a and b.
$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ is called the *closed interval* between a and b.
$(a, \infty) = \{x \in \mathbb{R} : x > a\}$
$[a, \infty) = \{x \in \mathbb{R} : x \geq a\}$
$(-\infty, a) = \{x \in \mathbb{R} : x < a\}$
$(-\infty, a] = \{x \in \mathbb{R} : x \leq a\}$
$(-\infty, \infty) = \{x \in \mathbb{R} : -\infty < x < \infty\} = \mathbb{R}$

Chapter 2

LIMITS AND CONTINUITY OF FUNCTIONS

2.1 Functions and their Graphs

Definition 2.1.1. Let X and Y be two sets. By a **function** f from X to Y, we mean a rule or relation which assigns to each x in X a unique element $f(x)$ in Y.

We often express the fact that f is a function from X to Y by writing

$$f : X \longrightarrow Y.$$

The set X is called the **domain** of f, and denoted by $Dom(f)$. The set of values $y = f(x)$ is called the **range** of f, and is denoted by $Ran(f)$. Equivalently, $Ran(f) = \{f(x) : x \in Dom(f)\}$ and is always a subset of Y might not equal Y. The set Y is called the **co-domain** of f. When we describe a function by writing a formula $y = f(x)$, we call x the *independent variable* and y the *dependent variable*. The set of all values for which f(x) is defined is the domain of f and the values $y = f(x)$ where $x \in Dom(f)$ is the range of f. The graph of the function f is the set of all points in the plane of the form $(x, f(x))$, where $x \in Dom(f)$.

Definition 2.1.2. We say that a function $f : X \longrightarrow Y$ is **one-to-one** or **injective** if, for every pair $x_1, x_2 \in X$, $x_1 \neq x_2$, implies that $f(x_1) \neq f(x_2)$ (or equivalently, if $f(x_1) = f(x_2)$, then $x_1 = x_2$).

Definition 2.1.3. A function $f : X \longrightarrow Y$ is **onto** or **surjective** if, every $y \in Y$ is of the form $f(x)$ for some $x \in X$. Equivalently, f is onto if $Ran(f) = Y$.

Definition 2.1.4. A function $f : X \longrightarrow Y$ is a **bijection** if it is one-to-one and onto.

Example 2.1.5. The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = 2x$ is one-to-one and onto, and hence it is a bijection.

Definition 2.1.6. If a function $f : X \longrightarrow Y$ is a **bijection** then there exists an inverse function $f^{-1} : Y \longrightarrow X$. For every $y \in Y$ there exists a unique element $x \in X$ defined by $x = f^{-1}(y)$.

2.1.1 Domain and Range of Functions

We now consider functions $f : X \longrightarrow Y$ for which X and Y subsets of real numbers $\mathbb{R}$. Such functions are said to be *real-valued*.

Example 2.1.7. For the function f with the formula $y = f(x) = 2x$, $Dom(f) = \mathbb{R}$ and $Ran(f) = \mathbb{R}$.

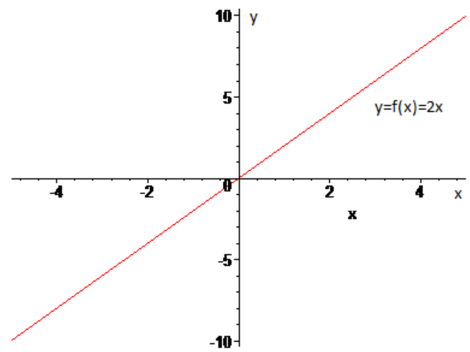

Figure 2.1: Graph of$y = f(x) = 2x$

Example 2.1.8. For the function f with the formula $y = f(x) = \sqrt{9 - x^2}$, $Dom(f) = \{x \in R : -3 \leq x \leq 3\} = [-3, 3]$ and $Ran(f) = \{y \in R : y \leq 3\} = (-\infty, 3]$.

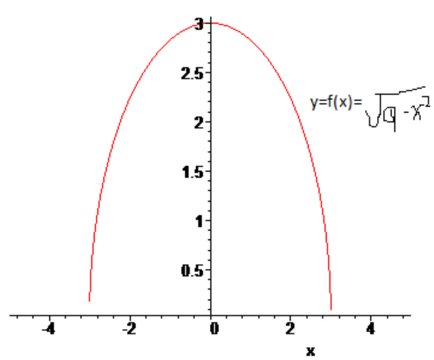

Figure 2.2: Graph of y=f(x)=$\sqrt{9 - x^2}$

Example 2.1.9. Let $y = f(x) = \frac{2-x}{x-1}$. Find $Dom(f)$ and $Ran(f)$.

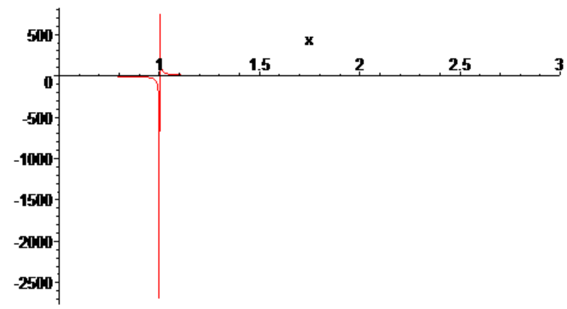

Figure 2.3: Graph of $y = f(x) = \frac{2-x}{x-1}$

It is easy to see that $Dom(f) = \{x \in \mathbb{R} : x \neq 1\} = \mathbb{R} - \{1\}$. Since $y = \frac{2-x}{x-1}$, then $x = \frac{y+2}{y+1}$, and the only value of $y$which will not occur is $y = -1$, for which the denominator is zero. Hence $Ran(f) = \mathbb{R} - \{1\}$

Composition of Functions

2.1.2 Some Elementary Functions

1. **Polynomials**. A real **polynomial** is a function whose domain is the set of real numbers, defined by

$$f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$$

where $a_0, a_1, ..., a_n$ are real constants(called the *coefficients* of f and $a_n \neq 0$). The integer n is called the degree of the polynomial, and denoted by $n = deg(f)$. Polynomials of degree $0, 1, 2, 3, 4, 5$ are called `constant`, `linear`, `quadratic, cubic, quartic` and `quintic`, respectively. For example, $f(x) = 4 - x^2 + 6x^8$ is a polynomial of degree 8, $g(x) = 2$ is a polynomial of degree 0 or a constant polynomial.

2. **Rational Functions**. A **rational function** $f(x)$ is a quotient of two polynomials. That is $f(x) = \frac{p(x)}{q(x)}$, where p and q are polynomials and $g(x) \neq 0$. For example, $f(x) = \frac{x^2-1}{x^3+1}$ is a rational function.

3. **Trigonometric and Inverse Trigonometric Functions**. The three basic trigonometric functions are *sine*, *cosine* and *tangent*. Reciprocals of these functions are $secant = 1/cosine$, $cosecant = 1/sine$, $cotangent = 1/tangent$. The inverses are *arcsine*, *arccosine*, *arctangent* and *arcotangent*.

4. **Exponential and Logarithmic Functions**. An exponential function is of the form $y = f(x) = a^x$, where x is a variable, and a is a constant. The domain of f is the real line $\mathbb{R} = (-\infty, \infty)$ and $Ran(f) = (0, \infty)$. When $a = e$, then $f(x) = e^x$.

The logarithm function is given by $y = f(x) = \log_a x$ and its domain is $(0, \infty)$ and its range is $\mathbb{R} = (-\infty, \infty)$. The constant a is referred to as the *base* of the logarithm function. The logarithm

function with base $a = e$ is called the *natural logarithm* or *Naperian logarithm*, and denoted by $\ln\ x$.
The following laws of logarithms are easy to derive from the laws of exponents:

- $\log_a xy = \log_a x + \log_a y$
- $\log_a \frac{x}{y} = \log_a x - \log_a y$
- $\log_a x^\alpha = \alpha \log_a x, \quad x\ any\ real\ number$
- $\log_a x = \frac{\log_b x}{\log_b a}$

These formulae hold for any positive $x, y, a, b (a \neq 1, b \neq 1)$.
5. **Hyperbolic Functions**. The four hyperbolic functions are hyperbolic cosine, hyperbolic sine, hyperbolic tangent and hyperbolic cotangent and are defined as follows:
$\cosh x = \frac{e^x + e^{-x}}{2}, \sinh x = \frac{e^x - e^{-x}}{2}, \tanh x = \frac{\sinh x}{\cosh x}, \coth x = \frac{\cosh x}{\sinh x}$.

2.2 Limits of Functions

2.2.1 Informal Definition of a Limit

Definition 2.2.1. (`Informal definition of a limit`) A function $f(x)$ is said to have a *limit* L as x approaches (or tends to) a if the values of $f(x)$ can be made as close as we like to L by taking x sufficiently close to a(but not equal to a). We use the two notations:
$\lim_{x \to a} f(x) = L$, read as " the limit of $f(x)$ as x tends to a is equal to L" or
$f(x) \longrightarrow L\ as\ x \longrightarrow a$, read as "f(x) tends to L as x tends to a".

Remarks. It is assumed that the domain of f includes an interval containing a, but not necessarily a itself. This definition is informal because phrases such as "`close as we want`" and "`close enough`" are imprecise; the meaning depends on the context. To a machinist manufacturing pistons, close enough may mean within a few thousandths of an inch. To an astronomer studying distant galaxies, close enough may mean within a few thousand light years. This definition should be clear enough, however, to enable us recognize and evaluate limits of specific functions.
Some limits are easy to compute but others are not so easy to evaluate. Limits of polynomials are found by substitution. Limits of (`many but not all`) rational functions can be found by substitution. Many situations require algebraic manipulation before applying the limits

Example 2.2.2. Find $\lim_{x \to 0} 3x^2 - 2x + 1$

Solution. As x tends to 0, it is evident that $f(x)$ approaches $3(0)^2 - 2(0) + 1 = 1$.

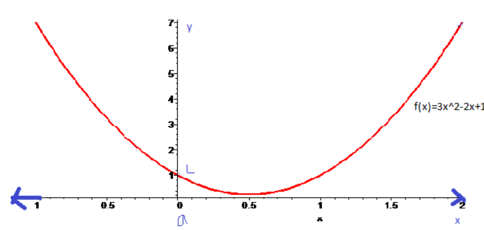

Figure 2.4: Limit of y=f(x)=3x$^2 - 2x + 1\ as\ x \longrightarrow 0$

Example 2.2.3. Find $\lim_{x\to -5} \frac{3x+1}{x-1}$.

Solution. As x tends to -5, it is evident that $f(x)$ approaches $\frac{3(-5)+1}{-5-1} = \frac{7}{3}$.
Remark. When we say that $f(x)$ has a limit L as x approaches a, we are really saying that we can ensure that the error $|f(x) - L|$ will be less than any allowed tolerance, no matter how small by taking x close enough to a (but not equal to a). It is traditional to use ϵ, the Greek letter "epsilon" for the error $|f(x) - L|$ and δ, the Greek letter "delta" for the difference $|x - a|$ that measures how close x is within that tolerance.

2.2.2 Formal $(\epsilon - \delta)$ Definition of a Limit

Definition 2.2.4. (**Formal(ϵ−δ) Definition of a Limit**) We say that a function $f(x)$ approaches the limit L as x approaches a, and write $\lim_{x\to a} f(x) = L$ if for every number $\epsilon > 0$, there exists a number $\delta > 0$, possibly depending on ϵ, such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

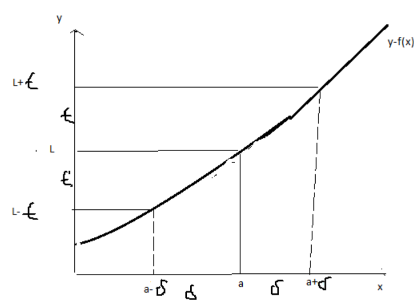

Figure 2.5: Limit of f(x) as x$\longrightarrow a$

Remark. Note that the formal definition of a limit does not tell us how to find the limit of a function, but it does enable us to verify a suspected limit is correct.
If $\lim_{x\to a} f(x) \neq L$, then there is an $\epsilon > 0$ such that for every $\delta > 0$ there is an $x \in \mathbb{R}$ with $0 < |x - a| < \delta$ and $|f(x) - L| \geq \epsilon$.

Example 2.2.5. Verify that $\lim_{x\to 2} 3x + 1 = 7$.

Solution. Let Let $\epsilon > 0$ be given. We want to find $\delta > 0$ so that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.
Now, $|f(x) - 7| = |3x + 1 - 7| = |3x - 6| = 3|x - 2| < \epsilon$ whenever $|x - 2| < \delta$. This means that $|x - 2| < \frac{1}{3}\epsilon$. Now, if we choose $\delta = \frac{1}{3}\epsilon$, we will have shown that $\lim_{x\to 2} 3x + 1 = 7$.
Note that the value $\delta = \frac{1}{3}\epsilon$ is not the only value that will make the implication $|f(x) - 7| = |3x + 1 - 7| = |3x - 6| = 3|x - 2| < \epsilon$ whenever $|x - 2| < \delta$ hold. Any smaller positive δ will do as well. The definition does not work for a "best" δ, just one that will work. We can thus choose $\delta \leq \frac{1}{3}\epsilon$ $\boxdot$.

Example 2.2.6. Verify that $\lim_{x\to 2} x^2 = 4$.

Solution. Let $\epsilon > 0$ be given. We want to find $\delta > 0$ so that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.
Now, $|f(x) - 4| = |x^2 - 4| = |(x + 2)(x - 2)| = |x + 2||x - 2|$.
We want the expression above to be less than ϵ. We can make the factor $|x - 2|$ as small as we wish/please by choosing δ properly, but we need to control the factor $|x + 2|$ so that it does not become too large. If we assume that $\delta \leq 1$(there is nothing special about 1. Another positive number could be used and not change the nature of the argument, the details would be different,however. But for convenience, it is customary to always select/choose δ to be less than or equal to 1) and require that $|x - 2| < \delta$, then we have

$$|x - 2| < 1 \Longrightarrow 1 < x < 3 \Longrightarrow 3 < x + 2 < 5 \Longrightarrow |x + 2| < 5.$$

Therefore $|f(x) - 4| < 5|x - 2|$ if $|x - 2| \leq 1$. But $5|x - 2| < \epsilon$ if $|x - 2| < \frac{\epsilon}{5}$.
Therefore, if we take $\delta = \min\{1, \frac{\epsilon}{5}\}$, then $|f(x) - 4| < 5|x - 2| < 5\frac{\epsilon}{5} = \epsilon$ if $|x - 2| \leq \delta$. This proves that $\lim_{x\to 2} x^2 = 4$.

Example 2.2.7. Use the $\epsilon - \delta$ definition of a limit to prove that $\lim_{x\to 3} x^2 = 9$.

Solution. Here $f(x) = x^2$ and $L = 9$, so that

$$|f(x) - L| = |x^2 - 9| = |(x + 3)(x - 3)| = |x + 3||x - 3| \quad ....(\star)$$

.
To make $|f(x) - L|$ small, one must control the size of $|x - 3|$. If by agreement we choose $\delta \leq 1$ and as a consequence the statement, "x is near 3" is to mean that x is restricted to the closed interval $[2, 4]$. That is, $2 \leq x \leq 4$. This information allows us to place bounds upon the factor $|x + 3|$. That is $|x + 3| < 7$. Thus $(\star)$ becomes

$$|f(x) - L| = |x^2 - 9| = |(x + 3)(x - 3)| = |x + 3||x - 3| < 7\delta < \epsilon \quad ....(\star\star),$$

where $\epsilon > 0$ and less than 1, is as small as you want it to be.
The inequality $(\star\star)$ tells us that if $\delta < \frac{\epsilon}{7}$, then it follows that $|x^2 - 9| < \epsilon$ whenever $|x - 3| < \delta$. So here if we choose $\delta = \min\{1, \frac{\epsilon}{7}\}$, then we are done.

Remark. The $\epsilon - \delta$ definition of a limit was developed around the $1800's$ and it resulted from the combined research developed by mathematicians Weierstrass, Bolzano and Cauchy.

2.3 Properties of Limits

Theorem 2.3.1. ***(Limit Theorems/Properties)****If* $\lim_{x\to a} f(x) = L_1$ *and* $\lim_{x\to a} g(x) = L_2$, *where* $L_1, L_2 \in \mathbb{R}$, *then*

1. $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L_1 + L_2$ *[Sum Rule]*

2. $\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L_1 - L_2$ *[Difference Rule]*

3. $\lim_{x\to a}[f(x).g(x)] = \lim_{x\to a} f(x). \lim_{x\to a} g(x) = L_1.L_2$ *[Product Rule]*

4. $\lim_{x\to a} k.f(x) = k \lim_{x\to a} f(x) = k.L_1$, *(for any number k)* *[Constant Multiple Rule]*

5. $\lim_{x\to a}[\frac{f(x)}{g(x)}] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L_1}{L_2}$ *if* $L_2 \neq 0$. *[Quotient Rule]*

Proof.(1) and (2) and (4) are easy to prove.
(3). Suppose $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$. Given $\epsilon > 0$ there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_1 \Longrightarrow |f(x) - L_1| < \frac{\epsilon}{2}$$

and

$$0 < |x - a| < \delta_2 \Longrightarrow |g(x) - L_2| < \frac{\epsilon}{2}.$$

Note also that if $\lim_{x\to a} g(x) = L_2$, then $0 < |x - a| < \delta_2 \Longrightarrow |g(x) - L_2| < \epsilon$. Letting $\epsilon = 1$ in this definition we have that $0 < |x - a| < \delta_2 \Longrightarrow |g(x) - L_2| < 1$ or $0 < |x - a| < \delta_2 \Longrightarrow -1 < g(x) - L_2 < 1$. Equivalently, $0 < |x - a| < \delta_2 \Longrightarrow -1 + L_2 < g(x) < 1 + L_2$ and hence

$$0 < |x - a| < \delta_2 \Longrightarrow |g(x)| < |1 + L_2| \leq 1 + |L_2| \ldots\ldots\ldots\ldots\ldots\ldots\ldots(\star)$$

Then

$$\begin{aligned} |f(x)g(x) - L_1L_2| &= |f(x)g(x) - L_1g(x) + L_1g(x) - L_1L_2| \\ &= |(f(x) - L_1)g(x) + L_1(g(x) - L_2)| \\ &\leq |(f(x) - L_1)g(x)| + |L_1(g(x) - L_2)| \\ &= |g(x)||f(x) - L_1| + |L_1||g(x) - L_2| \leq \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon. \end{aligned}$$

Therefore $\lim_{x\to a}[f(x).g(x)] = L_1.L_2$.

(5). Using (3), by using the fact that since $L_2 \neq 0 \Longrightarrow \lim_{x\to a} \frac{1}{g(x)} = \frac{1}{L_2}$, we have

$$\begin{aligned} |\tfrac{f(x)}{g(x)} - \tfrac{L_1}{L_2}| &= |\tfrac{f(x)}{g(x)} - \tfrac{L_1}{g(x)} + \tfrac{L_1}{g(x)} - \tfrac{L_1}{L_2}| \\ &= |\tfrac{1}{g(x)}(f(x) - L_1) + L_1(\tfrac{1}{g(x)} - \tfrac{1}{L_2}| \\ &\leq |\tfrac{1}{g(x)}||f(x) - L_1| + |L_1||\tfrac{1}{g(x)} - \tfrac{1}{L_2}| \\ &= \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon. \end{aligned}$$

Remark. Many situations require algebraic manipulation before the limit theorems can be applied.

Example 2.3.2. Compute the following limits
(a). $\lim_{x\to 3} \frac{x^2-9}{x-3}$
(b). $\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x}$
Solution.(a). The function $\frac{x^2-9}{x-3}$ is not defined when $x = 3$, since direct substitution gives $\frac{0}{0}$, which is undefined. That causes no difficulty, since the limit as x approaches 3 depends only on the values of x near 3 and excludes consideration of the values at $x = 3$ itself. To evaluate the limit, note that $x^2 - 9 = (x - 3)(x + 3)$. So for $x \neq 3$,

$$\frac{x^2 - 9}{x - 3} = \frac{(x - 3)(x + 3)}{x - 3} = x + 3.$$

As x approaches 3, $x+3$ approaches 6. Therefore

$$\lim_{x\to 3}\frac{x^2-9}{x-3}=6.$$

(b). Since the denominator approaches zero when taking the limit, we may not apply the Quotient Rule directly. However, if we first apply an algebraic trick, the limit may be evaluated. Multiply numerator and denominator by $\sqrt{x+4}+2$

$$\begin{aligned}\frac{\sqrt{x+4}-2}{x}\cdot\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2} &= \frac{(x+4)-4}{x(\sqrt{x+4}+2)}\\ &= \frac{x}{x(\sqrt{x+4}+2)}\\ &= \frac{1}{\sqrt{x+4}+2}\end{aligned}$$

Thus

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x} &= \lim_{x\to 0}\frac{1}{\sqrt{x+4}+2}\\ &= \frac{\lim_{x\to 0}1}{\lim_{x\to 0}(\sqrt{x+4}+2)}\\ &= \frac{1}{4}.\end{aligned}$$

Remarks. Note that if x approaches a and the values of $f(x)$ do not approach a specific number, then we say that the limit of $f(x)$ as x approaches a does not exist.

2.3.1 Right-Hand Limits and Left-Hand Limits

Sometimes the values of a function $f(x)$ tend to different limits as x approaches a number a from different sides. When this happens, we call the limit of f as x approaches a from the right the **right-hand limit** of f at a, and the limit of f as x approaches a from the left the **left-hand limit** of f at a. We use the notation

$\lim_{x\to a^+} f(x)$ ("the limit of f as x approaches a from the right").

$\lim_{x\to a^-} f(x)$ ("the limit of f as x approaches a from the left").

Definition 2.3.3. A function $f(x)$ has a limit L as x approaches a if and only if the right-hand and left-hand limits at a exist and are equal.
That is, $\lim_{x\to a} f(x)=L$ iff $\lim_{x\to a^+} f(x)=L$ and $\lim_{x\to a^-} f(x)=L$.

Example 2.3.4. Consider the function

$$f(x)=\begin{cases} x+2, & for\ x<1\\ 4 & for\ x=1\\ x+4 & for\ x>1\end{cases}$$

Show that f has no limit as $x\to 1$.

Solution. Clearly,

$$\lim_{x\to 1^-} f(x)=3\neq\lim_{x\to 1^+} f(x)=5.$$

Thus the two one-sided limits exits but are not equal. Thus $\lim_{x\to 1} f(x)$ does not exist.

Example 2.3.5. Graph the following function

$$f(x) = \begin{cases} \frac{x^2+x}{x}, & for\ x \neq 0 \\ 5 & for\ x = 0 \end{cases}$$

By inspection determine $\lim_{x\to 0} f(x)$, $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$.
By inspection

$$\lim_{x\to 0} f(x) = 1,\ \lim_{x\to 0^+} f(x) = 1,\ \lim_{x\to 0^-} f(x) = 1$$

Notice that $f(0) = 5$ and is not equal to $\lim_{x\to 0} f(x)$. Also note that the limit from the left and the limit from the right equal $\lim_{x\to 0} f(x)$.
Whenever a limit at a point exists, it is equal to the right-hand and left-hand limits. Also, whenever the right-hand and left-hand limits at a point both exist and are equal, the limit exists and is equal to the common value of the right-hand and left-hand limits.

Example 2.3.6. Use the $\epsilon - \delta$ definition of a limit to prove that the limit of a sum is the sum of the limits

$$\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L_1 + L_2.$$

Solution. By hypothesis, $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$, so that for a small number $\epsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|f(x) - L_1| < \frac{\epsilon}{2}\ when\ 0 < |x - a| < \delta_1$$

and

$$|g(x) - L_2| < \frac{\epsilon}{2}\ when\ 0 < |x - a| < \delta_2$$

Choose δ to be the smaller of δ_1 and δ_2, then using the triangle inequality, one can write

$$\begin{aligned} |(f(x) + g(x)) - (L_1 + L_2)| &= |(f(x) - L_1) + (g(x) - L_2)| \\ &\leq |f(x) - L_1| + |g(x) - L_2| \\ &\leq \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon\ when\ 0 < |x - a| < \delta \end{aligned}$$

Therefore $|(f(x) + g(x)) - (L_1 + L_2)| < \epsilon$ when $0 < |x - a| < \delta$, which implies that

$$\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L_1 + L_2. \quad \boxdot$$

Exercise. Use the $\epsilon - \delta$ definition of a limit to prove all the other limit theorems stated earlier.

Formal Definition of One-sided Limits

Definition 2.3.7. The limit of $f(x)$ as x approaches a `from the right` is the number L if the given $\epsilon > 0$ there exists $\delta > 0$ such that

$$a < x < a + \delta \ \ \textit{implies} \ \ |f(x) - L| < \epsilon.$$

Definition 2.3.8. The limit of $f(x)$ as x approaches a `from the left` is the number L if the given $\epsilon > 0$ there exists $\delta > 0$ such that

$$a - \delta < x < a \ \ \textit{implies} \ \ |f(x) - L| < \epsilon.$$

Remark. By comparing the two inequalities above, we can see the relation between the one-sided limits and the two-sided limit. If we subtract a from the δ-inequalities, they become

$$0 < x - a < \delta \Longrightarrow |f(x) - L| < \epsilon$$

and

$$-\delta < x - a < 0 \Longrightarrow |f(x) - L| < \epsilon.$$

Together, these two inequalities say the same thing as

$$0 < |x - a| < \delta \Longrightarrow |f(x) - L| < \epsilon,$$

which is the definition of the limit. In other words, $f(x)$ has a limit L at a if and only if the right-hand and left-hand limits of f at a exist and are equal.

2.3.2 Limits at Infinity and Infinite Limits

In this subsection, we study two types of limits
- limits at infinity, where x becomes arbitrarily large, positive or negative;
- infinite limits, which are not really limits at all but provide useful symbolism for describing the behaviour of functions whose values become arbitrarily large, positive or negative.

Limits at Infinity

Example 2.3.9. Consider the function $f(x) = \frac{x}{\sqrt{x^2+1}}$.

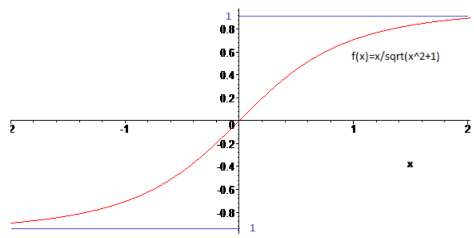

Figure 2.6: Limit of f(x) as x⟶±∞

The values of $f(x)$ seem to approach 1 as x takes on larger and larger positive values and -1 as x takes on negative values that get larger and larger in absolute value. This is equivalent to saying that $\lim_{x\to\infty} f(x) = 1$ and $\lim_{x\to-\infty} f(x) = -1$.
The graph of f conveys this limiting behaviour by approaching the horizontal lines $y = 1$ as $x \to \infty$ and $y = -1$ as $x \to -\infty$. These lines are called *horizontal asymptotes* of the graph. In general if a curve approaches a straight line as it recedes very far away from the origin, that line is called an *asymptote* of the curve.

Definition 2.3.10. (**Informal Definition of Limits at** ∞ **and** $-\infty$).
If the function f is defined on an open interval (a, ∞) and we can ensure that $f(x)$ is as close as we want to the limit L by taking x large enough, then we say that $f(x)$ *approaches the limit* L *as* x *approaches infinity*, and write

$$\lim_{x\to\infty} f(x) = L.$$

If the function f is defined on an open interval (a, ∞) and we can ensure that $f(x)$ is as close as we want to the limit L by taking x negative and large enough in absolute value, then we say that $f(x)$ *approaches the limit* L *as* x *approaches negative infinity*, and write

$$\lim_{x\to-\infty} f(x) = L.$$

Recall that the symbol ∞ does not represent a real number. We can not use it in arithmetic in the usual way, but we can use the phrase "*approaches* ∞" to mean "*becomes arbitrarily large positive*" and the phrase "*approaches* $-\infty$" to mean "*becomes arbitrarily large negative*".

Example 2.3.11. We can see that $\lim_{x\to\pm\infty}\frac{1}{x}=0$, and the x-axis is a horizontal asymptote of the graph of $y=f(x)=\frac{1}{x}$.

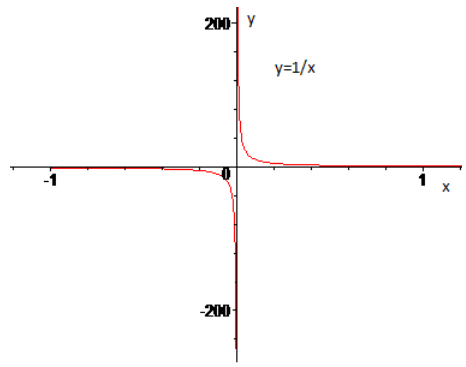

Figure 2.7: Limit of f(x) as x⟶±∞

We use algebraic manipulation/tricks to evaluate such limits.

Example 2.3.12. Evaluate $\lim_{x\to\infty}f(x)$ and $\lim_{x\to-\infty}f(x)$ for $f(x)=\frac{x}{\sqrt{x^2-1}}$.

Solution. Rewrite the expression for $f(x)$ as follows:

$$f(x)=\frac{x}{\sqrt{x^2+1}}=\frac{x}{\sqrt{x^2(1+\frac{1}{x^2})}}=\frac{x}{\sqrt{x}\sqrt{1+\frac{1}{x^2}}}=\frac{x}{|x|\sqrt{1+\frac{1}{x^2}}}=\frac{sgn(x)}{\sqrt{1+\frac{1}{x^2}}}$$

where $sgn(x)=\frac{x}{|x|}=\begin{cases}1, & x>0\\ -1, & x<0\end{cases}$

The factor $\sqrt{1+\frac{1}{x^2}}\longrightarrow 1$ as $x\longrightarrow\infty$ or $-\infty$. So $f(x)$ must have the same limit as $x\longrightarrow\pm\infty$ as does $sgn(x)$. Therefore $\lim_{x\to\infty}f(x)=1$ and $\lim_{x\to-\infty}f(x)=-1$.

2.3.3 Limits at Infinity for Rational Functions

The only polynomials that have limits at $\pm\infty$ are constant ones, $p(x)=c$. The situation is more interesting for rational functions. Recall that a rational function is a quotient of two polynomials. The following examples show how to render such a function in a form where its limits at infinity and negative infinity (if they exist) are apparent. The way to do this is to *divide the numerator and denominator by the highest power of x appearing in the denominator.* The limits of a rational function at infinity and negative infinity either both fail to exist or both exist and are equal.
We consider several cases

(a). **Numerator and Denominator of the same degree**. Divide the numerator and denominator by highest power of x, appearing in the denominator and apply the limit.

Example 2.3.13. Evaluate $\lim_{x\to\pm\infty} \frac{2x^2-x+3}{3x^2+5}$.

Solution. Divide the numerator and denominator by x^2 to have

$$\lim_{x\pm\infty} \frac{2x^2-x+3}{3x^2+5} = \lim_{x\to\pm\infty} \frac{2-1x+\frac{3}{x^2}}{3+\frac{5}{x^2}} = \frac{2-0+0}{3+0} = \frac{2}{3}.$$

(b). **Degree of Numerator less than degree of denominator**. Divide both numerator and denominator by the highest power of x in the denominator.

Example 2.3.14. Evaluate $\lim_{x\to\pm\infty} \frac{5x+2}{2x^3-1}$.

Solution. Divide the numerator and denominator by x^3 to have

$$\lim_{x\to\pm\infty} \frac{5x+2}{2x^3-1} = \lim_{x\to\pm\infty} \frac{\frac{5}{x^2}+\frac{2}{x^3}}{2-\frac{1}{x^3}} = \frac{0+0}{2-0} = 0.$$

(c). **Degree of Numerator is greater than the degree of Denominator**. Divide by the highest of x in the denominator.

Example 2.3.15. Evaluate $\lim_{x\to\pm\infty} \frac{2x^2-3}{7x+4}$.

Solution. Divide the numerator and denominator by x^2 to have

$$\lim_{x\to\infty} \frac{2x^2-3}{7x+4} = \lim_{x\to\pm\infty} \frac{2x-\frac{3}{x}}{7+\frac{4}{x}} = \pm\infty.$$

Infinite Limits

A function whose values grow arbitrarily large can sometimes be said to have an infinite limit. Since infinity is not a number, infinite limits are not really limits at all.

Example 2.3.16. $\lim_{x\to 0} \frac{1}{x^2} = \infty$.
Thus the limit does not exist(`DNE`) because the function $\frac{1}{x^2}$ becomes arbitrarily large near $x = 0$.

Changing Variables with Substitutions

Sometimes a change of variable can turn an unfamiliar expression into one whose limit we know how to find.

Example 2.3.17. Compute $\lim_{x\to\infty} \sin\frac{1}{x}$.

Solution. If we substitute $\theta = \frac{1}{x}$ then $\theta \to 0^+$ as $x \to \infty$. Thus

$$\lim_{x\to\infty} \sin\frac{1}{x} = \lim_{\theta\to 0^+} \sin\theta = 0.$$

2.3.4 The Sandwich or Squeeze Theorem

Theorem 2.3.18. *(**The Sandwich or Squeeze Theorem**) Suppose that*

$$g(x) \leq f(x) \leq h(x)$$

for all $x \neq a$ in some interval about a and that

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L.$$

Then $\lim_{x \to a} f(x) = L$.

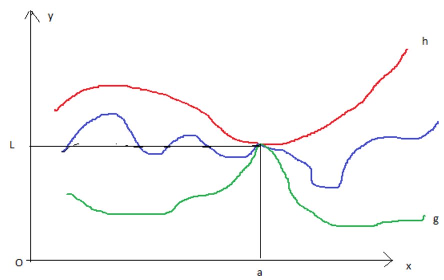

Figure 2.8: Sandwich or Squeeze Concept

Proof. Suppose $g(x) \leq f(x) \leq h(x)$, and $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$. Then given $\epsilon > 0$ there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_1 \Longrightarrow |g(x) - L| < \frac{\epsilon}{3}$$

and

$$0 < |x - a| < \delta_2 \Longrightarrow |h(x) - L| < \frac{\epsilon}{3}.$$

Letting $\delta = \min\{\delta_1, \delta_2\}$, we have

$$\begin{aligned} 0 < |x - a| < \delta \Longrightarrow |f(x) - L| &= |f(x) - g(x) + g(x) - L| \\ &\leq |f(x) - g(x)| + |g(x) - L| \\ &\leq |h(x) - g(x)| + |g(x) - L| \\ &= |h(x) - L - (g(x) - L)| + |g(x) - L| \\ &\leq |h(x) - L| + |g(x) - L| + |g(x) - L| \\ &= \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} = \epsilon. \end{aligned}$$

This proves that $\lim_{x \to a} f(x) = L$. $\boxdot$

Remark. The idea is that if the values of f are sandwiched between the values of of two functions that approach L, then the values of f approach L.

Example 2.3.19. The value of $\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$ can be found by the Sandwich Theorem. We sandwich $\frac{\sin\theta}{\theta}$ between the number 1 and a fraction that is known to approach 1 as $\theta \to 0$. This tells us that $\frac{\sin\theta}{\theta}$ approaches 1 as well.

2.3.5 Techniques of Evaluating Indeterminate Limits

(a). **Evaluating Indeterminate forms $\frac{0}{0}, \infty - \infty, \frac{\infty}{\infty}$ using algebraic tricks and change of variable Techniques**

Example 2.3.20. Find $\lim_{x\to 0} \frac{x}{\sqrt{1-x}-\sqrt{1+x}}$.

Solution. This is a "$\frac{0}{0}$" indeterminate form. We rationalize the denominator.

$$\lim_{x\to 0} \frac{x}{\sqrt{1-x}-\sqrt{1+x}} = \lim_{x\to 0} \frac{x(\sqrt{1-x}+\sqrt{1+x})}{(\sqrt{1-x}-\sqrt{1+x})(\sqrt{1-x}+\sqrt{1+x})} = -\lim_{x\to 0} \frac{\sqrt{1-x}+\sqrt{1+x}}{2} = -1$$

Example 2.3.21. Find $\lim_{x\to\infty}(\sqrt{x+1}-\sqrt{x})$.

Solution. This is an example of an "$\infty - \infty$" type. We multiply and divide by $(\sqrt{x+1}-\sqrt{x})$

$$(\sqrt{x+1}-\sqrt{x}) = (\sqrt{x+1}-\sqrt{x})\frac{(\sqrt{x+1}-\sqrt{x})}{(\sqrt{x+1}-\sqrt{x})} = \cdots = \frac{1}{(\sqrt{x+1}+\sqrt{x})} \to 0 \ as\ x \to \infty.$$

Example 2.3.22. Find $\lim_{x\to\infty} \sqrt{x}(\sqrt{x+1}-\sqrt{x})$.

Solution. Rationalization gives

$$\sqrt{x}(\sqrt{x+1}-\sqrt{x}) = \frac{\sqrt{x}}{(\sqrt{x+1}-\sqrt{x})}.$$

This is now an "$\frac{\infty}{\infty}$" form, and we divide the numerator and denominator by $\sqrt{x}$ to obtain

$$\sqrt{x}(\sqrt{x+1}-\sqrt{x}) = \frac{1}{(\sqrt{1+\frac{1}{x}}+1)} \to \frac{1}{2} \ \ as\ x \to \infty.$$

Example 2.3.23. Find $\lim_{x\to\frac{\Pi}{2}} \frac{\cos x}{\sin(\cos x)}$.

Solution. This is a "$\frac{0}{0}$" indeterminate form. We substitute $t = \cos x$. Then $t \to 0$ as $x \to \frac{\Pi}{2}$. Therefore

$$\lim_{x\to\frac{\Pi}{2}} \frac{\cos x}{\sin(\cos x)} = \lim_{t\to 0} \frac{t}{\sin t} = 1.$$

Example 2.3.24. Find $\lim_{x\to 0} \frac{\tan^{-1} x}{x}$.

Solution. This is a "$\frac{0}{0}$" indeterminate form. Let $t = \tan^{-1} x$ and $t \to 0$ as $x \to 0$. Therefore

$$\lim_{x\to 0} \frac{\tan^{-1} x}{x} = \lim_{t\to 0} \frac{t}{x}\tan t = \lim_{t\to 0} \frac{t}{\sin t}.\cos t = \lim_{t\to 0} \frac{t}{\sin t}.\lim_{t\to 0} \cos t = 1.$$

(b). **Evaluating Indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$ using L'Hôpital's Rule**
This rule is designed to deal with "$\frac{0}{0}$" and "$\frac{\infty}{\infty}$" indeterminate forms and it involves differentiation. This will be discussed in Chapter Two. Indeterminate forms like "$0.\infty$" and 1^∞ can be rewritten to look like $\frac{0}{0}$ or $\frac{\infty}{\infty}$, in which case L'Hôpital's Rule is applicable.

2.4 Continuous Functions

We first define the notion of continuity of a function in terms of limits and then a few equivalent definitions.

Definition 2.4.1. Suppose that a function f is defined on a subset A of $\mathbb{R}$ and $x \in A$. If $\lim_{x\to a} f(x)$ exists and if $\lim_{x\to a} f(x) = f(a)$, then we say that f is **continuous** at a.
A function f is **continuous on** A(its domain) if it is continuous at each point $a \in A$(its domain).

Remark. Briefly, continuity of a function f at a means that the limit of f at the point a is equal to the value of f at a. In other words, when x is close to a, then $f(x)$ is close to $f(a)$.

2.4.1 continuity Test and Points of Discontinuity

Continuity Test: For a function f to be continuous at a, it must satisfy the following three conditions:
1. f must be defined at a;

2. the limit, $\lim_{x\to a} f(x)$, must exist(i.e $< \infty$);

3. $\lim_{x\to a} f(x$ must equal $f(a)$.

If any one of these conditions is not satisfied, then f is not continuous at a.

Definition 2.4.2. If a function$f$ is not continuous at a, we say that it is **discontinuous** at a, or that a is a **discontinuity** of f or a **point of discontinuity** of f.

2.4.2 Formal ($\epsilon - \delta$) Definition of Continuity

Definition 2.4.3. (**Formal or $\epsilon - \delta$ definition of continuity**) A function f is continuous at $a \in A$ if for any $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in A$

$$|x - a| < \delta \ \ \textit{implies that} \ |f(x) - f(a)| < \epsilon.$$

Example 2.4.4. Verify that the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = 2x$ is continuous on $\mathbb{R}$.

Solution. Let $\epsilon > 0$ be given such that $|x - y| < \delta \Longrightarrow |f(x) - f(y)| < \epsilon$. That is

$$\begin{array}{rcl} |f(x) - f(y)| & < & \epsilon \\ |2x - 2y| & < & \epsilon \\ 2|x - y| & < & \epsilon \\ |x - y| & < & \frac{\epsilon}{2} \end{array}$$

So, if we choose $\delta \leq \frac{\epsilon}{2}$ will have shown that f is continuous on $\mathbb{R}$.

2.4.3 Types of Discontinuities

Remark. Note that in the formal definition of continuity of a function f, the number $\delta = \delta(x, \epsilon)$ is a function of the point x and ϵ. If $\delta = \delta(\epsilon)$ a function of ϵ only, we say that f is **uniformly continuous**.

Definition 2.4.5. If at a point $a \in A$ the limit $\lim_{x \to a} f(x)$ exists, but $f(a)$ either is not defined or $\lim_{x \to a} f(x) \neq f(a)$, then a is a **removable point of discontinuity**. In other words, f is discontinuous at a but can be re-defined at that *single point* so that it becomes continuous there.

Example 2.4.6. The function

$$f(x) = \{ \begin{array}{cc} x^3, & if\ x \neq 0 \\ 2, & if\ x = 0, \end{array}$$

is discontinuous at $a = 0$ and $\lim_{x \to 0} f(x) = 0$ so that $\lim_{x \to 0} f(x) \neq f(0)$. Thus the point 0 is a removable point of discontinuity. To remove it, define $f(0) = 0$.

Definition 2.4.7. If f is discontinuous at a point $a \in A$ and one-sided limits $f(a^-)$ and $f(a^+)$ both exists, but $f(a^-) \neq$f(a$^+$), then f is said to have a **discontinuity of the first kind** or **jump discontinuity**.

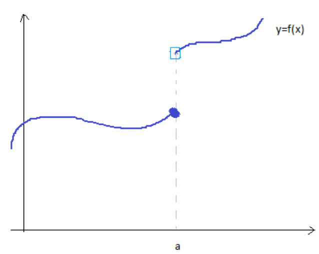

Figure 2.9: Jump Discontinuity at a

Example 2.4.8. The function

$$f(x) = \{ \begin{array}{cc} x, & if\ x < 1 \\ 3x^2 - 1, & if\ x \geq 1 \end{array}$$

is discontinuous at $a = 1$, the left-hand and right-hand limits of f at 1 both exists and $\lim_{x \to 1^-} f(x) = 1 \neq 2 = \lim_{x \to 1^+} f(x)$. Thus the point 1 is a jump discontinuity.

Definition 2.4.9. If at $a \in A$, at least one of the one-sided limits does not exist or is infinite, then we say that the point a is a discontinuity **discontinuity of the second kind**.

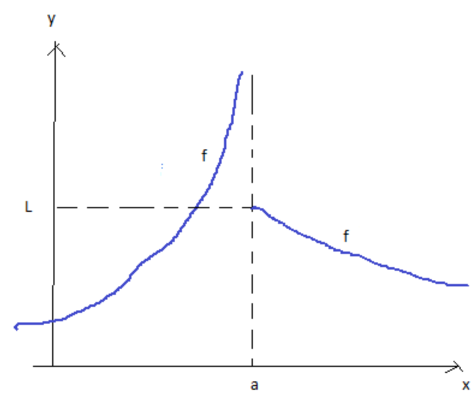

Figure 2.10: Discontinuity of the Second Kind at a

2.4.4 Right Continuity and Left Continuity

Definition 2.4.10. A function f is *continuous from the right*(*from the left*) at the point a if the following two conditions are satisfied:

1. $\lim_{x \to a^+} f(x) = f(a^+)$ $\Big(\lim_{x \to a^-} f(x) = f(a^-)\Big)$;
2. $f(a^+) = f(a)$ $\Big(f(a^-) = f(a)\Big)$.

Clearly a function f is continuous at a if it is both right continuous and left continuous at a.

Remark. Clearly, a function f is continuous at a point a provided that, roughly speaking, its graph has no *breaks*(*gaps* or *holes*) as it passes the point $(a, f(a))$. That is, f is continuous at a if we can draw the graph through $(a, f(a))$ without lifting our pencil from the paper.

Remark. Clearly the sine and cosine functions are continuous for every value of x tangent function is continuous where it is defined(its domain), polynomials are continuous at every point, rational functions are continuous wherever they are defined, the absolute-value function is continuous, all rational powers $x^{\frac{m}{n}}$ are continuous, exponential and logarithmic functions, hyperbolic functions are continuous.

2.4.5 Algebraic Properties of Continuous Functions

Theorem 2.4.11. *If the functions f and g are continuous at a point a, then the following functions are continuous at a:*

*1. **Sums**: $f+g$;*

*2. **Differences**: $f-g$;*

*3. **Products**: $f.g$;*

*4. **Constant multiples**: $k.f$;*

*5. **Quotients**: $\frac{f}{g}$ (provided that $g(a) \neq 0$).*

Proof. The proofs involve the use of limit theorems.

(a). $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = f(a)+g(a)$, so f + g is continuous at a.
The proofs to the other statements are done similarly, and are hence left as exercises.

2.5 Exercises and Some Solved Problems

1. Let $f, g, h : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $y = f(x) = \frac{1}{x-2}$, $y = g(x) = 3x-5$ and $y = h(x) = \sin(2x+1)$.
(a). Find

(i). the domain and range of f, g and h.

(ii). the composite functions $f \circ g, g \circ h$ and $g \circ f$.

(iii). $g^{-1}(x)$.

(b). Which of these functions is/are one-to-one, onto, many-to-one?

(c). Determine the domains of the functions

(i). $y = \sqrt{4-x^2}$ (ii). $y = \sqrt{x^2-16}$ (iii). $y = \frac{1}{x-3}$ (iv). $y = \frac{1}{x^2-4}$ (v). $y = \frac{x}{x^2+5}$

Solution.

(i). Since y must be real, $4-x^2 \geq 0$, or $x^2 \leq 4$. Thus $Dom(f)$ is the interval $-2 \leq x \leq 2$. It is easy to check that the range is $0 \leq y \leq 2$.

(ii). Here $x^2-16 \geq 0$, or $x^2 \geq 16$. Thus the domain consist of the intervals $x \leq -4$ and $x \geq 4$.

(iii). The function is defined for every real value except 3. Thus the domain is $\mathbb{R} - 3$.

(iv). The function is defined for $x \neq \pm 2$.

(v). Since $x^2+5 \neq 0$ for all x, the domain is the set of real numbers.

2. Find the following limits without using L'Hôpital's Rule.

(i). $\lim_{x\to 1} \frac{x-1}{\sqrt{x+3}-2}$. `Solution`: 4

(ii). $\lim_{x\to 0} \frac{(\frac{1}{-2+x})-(\frac{-1}{2})}{x}$. `Solution`: $\frac{-1}{4}$

(iii). $\lim_{x\to\infty} \frac{2+\sqrt{x}}{2-\sqrt{x}}$. `Solution`: -1

3. (a). State the formal definition of a limit of a function $f(x)$ as x approaches a.

(b). Use the formal definition of a limit to verify the indicated limit in each case.

(i). $\lim_{x\to 1} \frac{1}{x+1} = \frac{1}{2}$.

(ii). $\lim_{x\to 5} x^2 = 25$.

(iii). $\lim_{x\to 2} 5 - 2x = 1$.

(c). Evaluate the following limits.

(i). $\lim_{x\to -\infty} \frac{\cos\frac{1}{x}}{1+\frac{1}{x}}$.

(ii). $\lim_{x\to\infty} (\frac{1}{x})^{\frac{1}{x}}$.

(iii). $\lim_{x\to\pm\infty} (3 + \frac{2}{x})(\cos\frac{1}{x})$.

(d). Use the formal definition of a limit to verify that $\lim_{x\to 2} x^2 + 3x = 10$.

Solution Let $\epsilon > 0$ be chosen. We must produce a $\delta > 0$ such that, whenever $0 < |x-2| < \delta$ then $|(x^2+3x) - 10| < \epsilon$. That is, $|(x-2)(x+5)| < \epsilon$. Let $\delta \leq 1$. Then $|x-2| < 1$ implies that $1 \leq x \leq 3$ and hence $|x+5| \leq 8$. Thus $0 < |x-2| < \delta$ then $8|(x-2)| < \epsilon$ and thus $|x-2| < \frac{\epsilon}{8}$. Hence, if we take δ to be the minimum of 1 and $\frac{\epsilon}{8}$, then whenever $0 < |x-2| < \delta$, $|(x^2+3x) - 10| < 8\delta \leq \epsilon$.

4. (a). State the Sandwich/Squueze Theorem.

(b). Given that $3 - x^2 \leq f(x) \leq 3 + x^2$ for all $x \neq 0$, find $\lim_{x\to 0} f(x)$.

5. (a). Evaluate the limit or explain why it does not exist.

(i). $\lim_{x\to 4}(x^2 - 4x + 1)$.

(ii). $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$.

(iii). $\lim_{x\to 3} \frac{x^2-6x+9}{x^2-9}$.

(iv). $\lim_{x\to 0} \frac{1}{3+2^{\frac{1}{x}}}$.

Solution.

- Note that as $x \to 0^-$, $\frac{1}{x} \to -\infty$, and $2^{\frac{1}{x}} \to 0$, and $\lim_{x\to 0} \frac{1}{3+2^{\frac{1}{x}}} = \frac{1}{3}$.
- As $x \to 0^+$, $\frac{1}{x} \to \infty$, and $2^{\frac{1}{x}} \to \infty$, and $\lim_{x\to 0} \frac{1}{3+2^{\frac{1}{x}}} = 0$.

Thus $\lim_{x\to 0} \frac{1}{3+2^{\frac{1}{x}}}$ does not exist.

(b). (i). State the formal definition of continuity of a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ at the point $x_0 \in \mathbb{R}$.

(ii). Prove that the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = x^2$ is continuous at $x_0 \in \mathbb{R}$.

(c). Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} x, & x < 1 \\ 5 - x, & x \geq 1 \end{cases}$$

(i). Sketch the graph of f.

(ii). Is f continuous at $x = 1$? If not, state the nature of discontinuity of f at $x = 1$.

(d). Show that at $x = 0$, the function $f(x) = \frac{1}{3^{\frac{1}{x}}+1}$ has a jump discontinuity.

Chapter 3

DIFFERENTIATION OF FUNCTIONS OF A SINGLE VARIABLE

Two fundamental problems are considered in calculus:

• The problem of slopes is concerned with finding the slope of (the tangent line to) a given curve at a given point on the curve.

• The problem of areas is concerned with finding the area of a plane region bounded by curves and straight lines.

The solution of the problem of slopes is the subject of differential calculus. As we will see, it has many applications in mathematics and other disciplines.

The problem of areas is the subject of integral calculus, which we begin in Chapter 4.

3.1 Tangent Lines and Their Slopes

3.1.1 Tangent Lines

Conditions for Tangency to a Curve at a Point

Let $\mathfrak{C}$ be the graph of a function $y = f(x)$ and let P be the point (x_0, y_0) on $\mathfrak{C}$, so that $y_0 = f(x_0)$.

What do we mean when we say that the line L is tangent to C at P?

Note that a tangent line L should have the "same direction" as the curve does at the point of tangency P.

A reasonable definition of tangency can be stated in terms of limits. If Q is a point on $\mathfrak{C}$ different from P, then the line through P and Q is called a **secant line** to the curve. This line rotates around P as Q moves along the curve. If L is a line through P whose slope is the limit of the slopes of these secant lines PQ as Q approaches P along $\mathfrak{C}$, then we will say that L **is tangent to** $\mathfrak{C}$ **at** P.

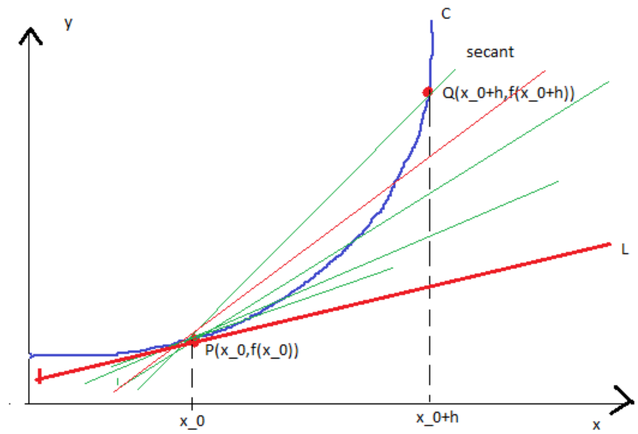

Figure 3.1: Secant lines PQ approach tangent line L as Q approaches P along the curve C

The slope of the line PQ is

$$\frac{f(x_0+h)-f(x_0)}{h}.$$

Note that h can be positive or negative, depending on whether Q is to the right or left of P.

Definition 3.1.1. Suppose f is continuous at $x = x_0$ and that

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} = m,$$

exists. Then the straight line having slope m and passing through the point $P = (x_0, f(x_0))$ is called the **tangent line** (or simply the **tangent**) to the graph of $y = f(x)$ at P. An equation of this tangent is

$$y = m(x - x_0) + y_0.$$

Example 3.1.2. Find an equation of the tangent line to the curve $y = x^2$ at the point $(1, 1)$.

Solution. Here $f(x) = x^2, x_0 = 1$, and $y_0 = f(1) = 1$. The slope of the required tangent is

$$\begin{aligned} m = \lim_{h\to 0} \tfrac{f(1+h)-f(1)}{h} &= \lim_{h\to 0} \tfrac{(1+h)^2-1}{h} \\ &= \lim_{h\to 0} \tfrac{1+2h+h^2-1}{h} \\ &= \lim_{h\to 0} \tfrac{2h+h^2}{h} \\ &= \lim_{h\to 0}(2+h) = 2. \end{aligned}$$

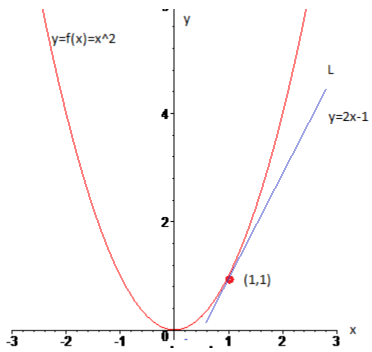

Figure 3.2: Secant lines PQ approach tangent line L as Q approaches P along the curve C

Accordingly, the equation of the tangent line at $(1, 1)$ is $y = 2(x - 1) + 1$ or $y = 2x - 1$.

3.1.2 The Slope of a Curve

Definition 3.1.3. The slope of a curve $\mathfrak{C}$ at a point P is the slope of the tangent line to $\mathfrak{C}$ at P if such a tangent line exists. In particular, the slope of the graph of $y = f(x)$ at the point x_0 is

$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Example 3.1.4. Find the slope of the curve $y = x^2$ at the point $x = -2$.

Solution. If $x = -2$, then $y = 4$, so the slope is

$$\begin{aligned} m &= \lim_{h\to 0} \tfrac{f(x_0+h)-f(x_0)}{h} \\ &= \lim_{h\to 0} \tfrac{(-2+h)^2-4}{h} \\ &= \lim_{h\to 0} \tfrac{4-4h+h^2-4}{h} \\ &= \lim_{h\to 0} \tfrac{-4h+h^2}{h} = \lim_{h\to 0}(-4+h) = -4. \end{aligned}$$

The **average rate of change** in a quantity over a period of time is the amount of change divided by the time it takes. For instance, average speed is distance traveled divided by the elapsed time, say, in km/hour,etc.

3.2 The Derivative

Definition 3.2.1. The **derivative** of a function f is another function f' (read as "f prime") defined by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

at all points x for which the limit exists (i.e., is a finite real number). If $f'(x)$ exists, we say that f is **differentiable** at x otherwise, it is said to be **non-differentiable** at x.

Remark. Note that the domain of f' is the subset of $dom(f)$ where f' is defined. That is, The domain of the derivative f' is the set of numbers x in the domain of f where the graph of f has a non-vertical tangent line. $Dom(f')$ of f' may be smaller than $dom(f)$ because it contains only those points in $dom(f)$ at which f is differentiable. Values of x in $dom(f)$ where f is not differentiable and that are not endpoints of $dom(f)$ are **singular points** of f.

Remark. The value of the derivative of f at a particular point x_0 can be expressed as a limit in either of two ways:

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}.$$

In the second limit x_0+h is replaced by x, so that $h = x-x_0$ and $h \to 0$ is equivalent to $x \to x_0$. The process of calculating the derivative f' of a given function f is called **differentiation**. A function is **differentiable on a set** S if it is differentiable at every point x in S. Polynomial functions are differentiable, as are rational functions and trigonometric functions (in their domains). Composites of differentiable functions are differentiable, and so are sums, differences, products, powers, and quotients of differentiable functions, where defined. Derivatives are the functions we use to measure the rates at which things change.

3.2.1 Right Derivative and Left Derivative

$f'_+(a) = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$ and $f'_-(a) = \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h}$

3.2.2 Leibniz Notation

Because functions can be written in different ways, it is useful to have more than one notation for derivatives. If $y = f(x)$, we can use the dependent variable y to represent the function, and we can denote the derivative of the function with respect to x in any of the following ways:

$$D_x y = y' = \frac{dy}{dx} = \frac{d}{dx} f(x) = f'(x) = D_x f(x) = Df(x).$$

Often the most convenient way of referring to the derivative of a function given explicitly as an expression in the variable x is to write $\frac{d}{dx}$ (read ad "d dx") in front of that expression. The symbol $\frac{d}{dx}$ is a *differential operator* and should be read "the derivative with respect to x of . . ." The

notations $\frac{dy}{dx}$ and $\frac{d}{dx}f(x)$ are called Leibniz notations for the derivative, after Gottfried Wilhelm Leibniz (1646-1716), one of the creators of calculus, who used such notations.
The main ideas of calculus were developed independently by Leibniz and Isaac Newton (1642-1727). Newton used notations similar to the prime y' notations.

The Newton quotient $\frac{[f(x+h)-f(x)]}{h}$, whose limit we take to determine the derivative $\frac{dy}{dx}$ (read ad "dy dx"), can be written in the form $\frac{\Delta y}{\Delta x}$, where $\Delta y = f(x+h) - f(x)$ is the increment of y and $\Delta x = (x+h) - x = h$ is the corresponding increment in x as we pass from the point $(x, f(x))$ to the point $(x+h, f(x+h))$ on the graph of f.

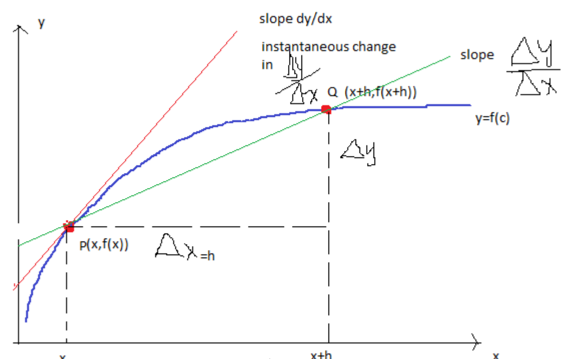

Figure 3.3: $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

3.2.3 Differentiability and Continuity

Recall that we have defined differentiability of a function $f(x)$ at x_0 in terms of a limit. If this limit does not exists, then we said that f is non-differentiable at $x = x_0$. Geometrically, the non-differentiability of f at $x = x_0$ can manifest itself in several different ways. First of all, the graph of $f(x)$ could have no tangent line at $x = x_0$. Secondly, the graph could have a vertical tangent line at $x = x_0$.
Closely related to the concept of differentiability is the concept of continuity. Note that every differentiable function at $x = a$ is continuous there. However, a function may be continuous at $x = a$ but still not be differentiable there. For instance, the absolute value function $f(x) = |x|$ is continuous on $\mathbb{R}$ but not differentiable at $x = 0$.

Theorem 3.2.2. ***(Differentiability implies continuity)***. *If a function f is differentiable at x, then f is continuous at x.*

Proof. Since f is differentiable at x, we know that

$$\lim_{h \to \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

exists. But using the limit theorems, we have

$$\begin{aligned} \lim_{h \to \to 0} \Big(f(x+h) - f(x) \Big) &= \lim_{h \to \to 0} \left(\tfrac{f(x+h)-f(x)}{h} \right)(h) \\ &= (f'(x))(0) = 0. \end{aligned}$$

This is equivalent to

$$\lim_{h \to \to 0} f(x+h) = f(x),$$

which says that f is continuous at x. ⊡

3.3 Differentiation Rules

If every derivative had to be calculated directly from the definition(Method of First Principles), calculus would indeed be a painful subject. In this section we show how to differentiate functions rapidly-without having to apply the definition each time. We now give several examples of the calculation of derivatives algebraically from the definition of derivative(by **Method of First Principles**). Some of these are the basic building blocks from which more complicated derivatives can be calculated later.

Theorem 3.3.1. ***(Rule 1: Derivative of a Constant and a Linear Function)***
If $f(x) = ax + b$, then $f'(x) = a$.

Proof. Using the definition (First Principles)

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \tfrac{f(x+h)-f(x)}{h} \\ &= \lim_{h \to 0} \tfrac{a(x+h)+b-(ax+b)}{h} \\ &= \lim_{h \to 0} \tfrac{ah}{h} = a. \end{aligned}$$ ⊡

A consequence of this result is that the derivative of a constant function is zero. That is, if $g(x) = c$ (*constant*), then $g'(x) = 0$.

Theorem 3.3.2. ***(Rule 2: Derivative of a Power Function)***
If $f(x) = x^r$, then $f'(x) = rx^{r-1}$, when x^r makes sense as a real number.

Proof. Let $r = n$, a positive integer. Note that

$$a^n - b^n = (a-b)\Big[a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1}\Big].$$

If $f(x) = x^n, a = x + h$ and $b = x$, then $a - b = h$, and,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \tfrac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{h\left[\overbrace{(x+h)^{n-1} + (x+h)^{n-2}x + (x+h)^{n-3}x^2 + \cdots + x^{n-1}}^{n \text{ terms each with limit } x^{n-1} \text{ as } h \text{ approaches zero}}\right]}{h} \\
&= \lim_{\to 0} \left[\overbrace{x^{n-1} + x^{n-1} + x^{n-1} + \cdots + x^{n-1}}^{n \text{ terms}}\right] \\
&= nx^{n-1}. \qquad \boxdot
\end{aligned}$$

Theorem 3.3.3. ***(Rule 3: Constant Multiple Rule)***
If f is a differentiable function of x and k is a constant, the

$$\frac{d}{dx}(k.f) = k.\frac{df}{dx}.$$

Proof. Easy and left as an exercise.

Theorem 3.3.4. ***(Rule 4: Sum and Difference Rule)***
If f and g are a differentiable functions of x then their sum and difference are differentiable at every point where f and g are both differentiable. At such points,

$$\frac{d}{dx}(f \pm g) = \frac{df}{dx} \pm \frac{dg}{dx}.$$

Proof. Exercise.
This rule can also be generalized to a finite set of functions.

Second and Higher Order Derivatives The derivative

$$y' = \frac{dy}{dx}$$

is the *first derivative* of y with respect to x. If the first derivative is also a differentiable function, then its derivative

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d^2y}{dx^2},$$

is called the *second derivative* of f with respect to x. If y'' (" y double prime") is differentiable, then its derivative

$$y''' = \frac{dy''}{dx} \text{ ("}y \text{ \textit{triple prime}")}$$

is the *third derivative* of f with respect to x and so on. Thus

$$y^{(n)} = \frac{d}{dx}y^{(n-1)} \text{ ("}y \text{ \textit{super} } n\text{")}$$

denotes the *nth derivative*of y with respect to x.

Theorem 3.3.5. ***(Rule 5: The Product Rule)***
The product of two differentiable functions f and g of x is differentiable and

$$\frac{d}{dx}(f.g) = f\frac{dg}{dx} + g\frac{df}{dx}.$$

Proof. Let f and g of x be differentiable functions of x. Then

$$\begin{aligned}
(fg)'(x) &= \lim_{h\to 0} \tfrac{f(x+h)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\to 0} \tfrac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\to 0} \left[[\tfrac{f(x+h)-f(x)}{h}]g(x+h) + f(x)[\tfrac{g(x+h)-g(x)}{h}] \right] \\
&= f'(x)g(x) + f(x)g'(x) \qquad \boxdot
\end{aligned}$$

Note that to get to the last line of the proof, we have used the fact that g is continuous(since it is differentiable and hence $g(x+h) \to g(x)$ as $h \to 0$).

Theorem 3.3.6. ***(Rule 5: The Quotient Rule)***
At a point where $g \neq 0$, the quotient $y = \frac{f}{g}$ of two differentiable functions of x is differentiable and

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}.$$

Proof.Let f and g of x be differentiable functions of x and suppose $g(x) \neq 0$. Then

$$\begin{aligned}
(\tfrac{f}{g})'(x) &= \lim_{h\to 0} \tfrac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} \\
&= \lim_{h\to 0} \tfrac{g(x)f(x+h-f(x)g(x+h))}{hg(x+h)g(x)} \\
&= \lim_{h\to 0} \tfrac{g(x)f(x+h)-g(x)f(x)+g(x)f(x)-f(x)g(x+h)}{hg(x+h)g(x)} \\
&= \lim_{h\to 0} \tfrac{g(x)[\frac{f(x+h)-f(x)}{h}]-f(x)[\frac{g(x+h)-g(x)}{h}]}{g(x+h)g(x)} \\
&= \tfrac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2} \qquad \boxdot
\end{aligned}$$

Note that to get to the last line of the proof, we have used the fact that g is continuous(since it is differentiable and hence $g(x+h) \to g(x)$ as $h \to 0$).

Theorem 3.3.7. ***(Rule 6: The Chain Rule)***
If f $f(u)$ is differentiable at $u = g(x)$, and $g(x)$ is differentiable at x, then the composite function $f \circ g(x) = f(g(x))$ is differentiable at x, and $(f \circ g)'(x) = f'(g(x))g'(x)$.

Proof 1. Suppose that f is differentiable at the point $u = g(x)$ and that g is differentiable at x. Let the function $E(k)$ be defined by

$$\begin{aligned} E(0) &= 0, \\ E(k) &= \tfrac{f(u+k)-f(u)}{k} = f'(u),\ if\ k \neq 0. \end{aligned}$$

By the definition of derivative, $\lim_{k\to 0} E(k) = f'(u) - f'(u) = 0 = E(0)$, and so $E(k)$ is continuous at $k = 0$. Also, whether $k = 0$ or not, we have

$$f(u + k) - f(u) = (f'(u) + E(k))k.$$

Now, put $u = g(x)$ and $k = g(x + h) - g(x)$, so that $u + k = g(x + h)$, and obtain

$$f(g(x + h)) - f(g(x)) = (f'(g(x) + E(k))(g(x + h) - g(x)).$$

Since g is differentiable at x, $\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = g'(x)$.
Also, g is continuous at x, so $\lim_{h\to 0} k = \lim_{h\to 0}(g(x + h) - g(x)) = 0$. Since E is continuous at 0, $\lim_{h\to 0} E(k) = \lim_{k\to 0} E(k) = E(0) = 0$.
Hence

$$\begin{aligned} [f(g(x))]' &= \lim_{h\to 0} \tfrac{f(g(x+h))-f(g(x))}{h} \\ &= \lim_{h\to 0} \Big(f'(g(x) + E(k))\Big) \tfrac{g(x+h)-g(x)}{h} \\ &= \Big(f'(g(x)) + 0\Big) g'(x) \\ &= f'(g(x))g'(x). \qquad \boxdot \end{aligned}$$

Proof 2.

$$\begin{aligned} [f(g(x))]' &= \lim_{h\to 0} \tfrac{f(g(x+h))-f(g(x))}{h} \\ &= \lim_{h\to 0} \tfrac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)} . \tfrac{(g(x+h)-g(x))}{h} \\ &= \lim_{h\to 0} \tfrac{f(u+k)-f(u)}{k} . \tfrac{(g(x+h)-g(x))}{h} \\ &= f'(u)g'(x) = f'(g(x))g'(x). \qquad \boxdot \end{aligned}$$

`Idea of the proof`. Introduce $g(x+h)-g(x)$in the numerator and denominator and let $g(x) = u$ and $g(x + h) = u + k$. Note k$\to 0$ *as* $h \to 0$.

Remark. The Chain Rule tells us how to differentiate *composites* of functions whose derivatives we already know. If $y = f(u)$, where $u = g(x)$, then $y = f(g(x))$ and:
at u, y is changing $\frac{dy}{du}$ times as fast as u is changing;

at x, u is changing $\frac{du}{dx}$ times as fast as x is changing.
Therefore, at x, $y = f(u) = f(g(x))$ is changing $\frac{dy}{du}\frac{du}{dx}$ times as fast as x is changing. That is

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx},$$

where $\frac{dy}{du}$ is evaluated at $u = g(x)$.

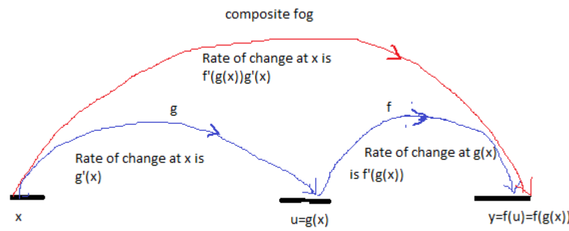

Figure 3.4: Chain Rule

Example 3.3.8. Find the derivative of $y = \sqrt{x^2+1}$

Solution Here $y = f(g(x))$, where $f(u) = \sqrt{u}$ and $g(x) = x^2 + 1$. Since $f'(u) = \frac{1}{2\sqrt{u}}$, $g'(x) = 2x$, then by the Chain Rule

$$\begin{aligned} \frac{dy}{dx} = \frac{d}{dx}(f(g(x))) &= f'(g(x))g'(x) \\ &= \frac{1}{2\sqrt{g(x)}}(2x) \\ &= \frac{1}{2\sqrt{x^2+1}}(2x) \\ &= \frac{x}{2\sqrt{x^2+1}} \end{aligned}$$

Usually, when applying the Chain Rule, we do not introduce symbols to represent the functions being composed, but rather just proceed to calculate the derivative of the "`outside`" function and then multiply by the derivative of whatever is "`inside`." You can say to yourself: "the derivative of f of something is f' of that thing, multiplied by the derivative of that thing." This is called the "`Inside-Outside`" Rule.

Example 3.3.9. Find the derivatives of
(a). $y = (8x-3)^5$
(b). $y = f(xt) = \sin(x^2-4)$
(c). $y = \left(3x + \frac{1}{(2x+1)^3}\right)^{\frac{1}{4}}$
(d). $y = \cos^2(3x)$

Solution.

(a). Let $u = 7x - 3$. Thus $y = u^{10}$. Clearly, $\frac{dy}{du} = 10y^9$ and $\frac{du}{dx} = 7$. Thus by the Chain Rule $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 10u^9(7) = 10u^9 = 70(7x-3)$.

(b). Let $u = x^2 - 4$. Then $y = \sin u$ and $\frac{dy}{du} = \cos u$ and $\frac{du}{dx} = 2x$. Thus $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \cos u.(2x) = 2x\cos(x^2-4)$.

Repeated Use. We sometimes have to use the Chain Rule two or more times to get the job done.

(c).

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{4}(3x + \frac{1}{(2x+1)^3})^{-\frac{3}{4}}\frac{d}{dx}(3x + \frac{1}{(2x+1)^3}) \\ &= \frac{1}{4}(3x + \frac{1}{(2x+1)^3})^{-\frac{3}{2}}(3 - \frac{3}{(2x+1)^4})\frac{d}{dx}(2x+1) \\ &= \frac{3}{4}(1 - \frac{2}{(2x+1)^4})(3x + \frac{2}{(2x+1)^3})^{-\frac{3}{4}}. \end{aligned}$$

(d).

$$\begin{aligned}
\tfrac{d}{dx}\cos^2(3x) &= 2\cos 3x.\tfrac{d}{dx}(\cos 3x) \quad (\textit{Power and Chain Rule}) \\
&= 2\cos 3x(-\sin 3x)\tfrac{d}{dx}(3x) \quad (\textit{Chain Rule again}) \\
&= 2\cos 3x(-\sin 3x)(3) \\
&= -6\cos 3x\sin 3x.
\end{aligned}$$

Note. The choice of which rule to use in solving a differentiation problem can make a difference in how much work you have to do.

3.4 Derivatives of Trigonometric and Inverse Trigonometric Functions

Trigonometric functions are important because so much of the phenomena we want information about are periodic(heart rhythms, earthquakes, tides, weather,etc). Continuous periodic functions can always be expressed in terms of sines and cosines, so the derivatives of sines and cosines play an important role in describing important changes.

Some Special Limits

(1). $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$

(2). $\lim h \to 0\frac{\cos h - 1}{h} = 0.$

3.4.1 Derivative of Sine and Cosine

Theorem 3.4.1. $\frac{d}{dx}\sin x = \cos x.$

Proof. We use the Definition/First Principles.

$$\begin{aligned}
\tfrac{d}{dx}\sin x &= \lim_{h\to 0}\tfrac{\sin(x+h) - sinx}{h} \\
&= \lim_{h\to 0}\tfrac{\sin x\cos h + \cos x\sin h - sinx}{h} \\
&= \lim_{h\to 0}\tfrac{\sin x(\cos h - 1) + \cos x\sin h}{h} \\
&= \lim_{h\to 0}\sin x.\lim_{h\to 0}\tfrac{(\cos h - 1)}{h} + \cos x.\lim_{h\to 0}\tfrac{\sin h}{h} \\
&= \sin x(0) + (\cos x).(1) \\
&= \cos x.
\end{aligned}$$

Theorem 3.4.2. $\frac{d}{dx}\cos x = -\sin x.$

Proof. We use the Definition/First Principles.

$$\begin{aligned}
\frac{d}{dx}\cos x &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} \\
&= \lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h} \\
&= \lim_{h\to 0}\frac{\cos x(\cos h-1)-\sin x\sin h}{h} \\
&= \lim_{h\to 0}\cos x.\lim_{h\to 0}\frac{(\cos h-1)}{h}-\sin x.\lim_{h\to 0}\frac{\sin h}{h} \\
&= \cos x(0)-(\sin x).(1) \\
&= -\sin x.
\end{aligned}$$

Recall that $\sin(\frac{\Pi}{2}-x)=\cos x$ and $\cos(\frac{\Pi}{2}-x)=\sin x$. Using the Chain Rule we have: $\frac{d}{dx}\cos x = \frac{d}{dx}\sin(\frac{\Pi}{2}-x)=\cos(\frac{\Pi}{2}-x)(-1)=-sinx.$

3.4.2 Derivatives of other Basic Trigonometric Functions

Because $\sin x$ and $\cos x$ are differentiable functions of x, the related functions

$$\tan x=\frac{\sin x}{\cos x} \qquad \sec x=\frac{1}{\cos x}$$

$$\cot x=\frac{\cos x}{\sin x} \qquad \csc x=\frac{1}{\sin x}$$

are differentiable at every value of x at which they are defined(i.e. their domain). Their derivatives, calculated using the Quotient Rule are:

$$\frac{d}{dx}\tan x=\sec^2 x \qquad \frac{d}{dx}\sec x=\sec x\tan x$$

$$\frac{d}{dx}\cot x=\csc^2 x \qquad \frac{d}{dx}\csc x=-\csc x\cot x$$

Example 3.4.3. Find $\frac{dy}{dx}$ if $y=\tan x$.

Solution.

$$\begin{aligned}
\frac{d}{dx}\tan x &= \frac{d}{dx}\frac{\sin x}{\cos x}=\frac{\cos x\frac{d}{dx}(\sin x)-\sin x\frac{d}{dx}(\cos x)}{\cos^2 x} \\
&= \frac{\cos x-\sin x(-\sin x)}{\cos^2 x}=\frac{\cos^2 x+\sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x}=\sec^2 x.
\end{aligned}$$

Example 3.4.4. Find $\frac{dy}{dx}$ if $y=\sec x$.

Solution.

$$\frac{d}{dx}\sec x=\frac{d}{dx}(\frac{1}{\cos x})=\frac{0.\cos x-1(-\sin x)}{\cos^2 x}=\frac{\sin x}{\cos^2 x}=\frac{\sin x}{\cos x}\frac{1}{\cos x}=\tan x\sec x.$$

Exercises.
1. Find the derivatives of the following functions

(a). $y = 3x + \cot(\frac{x}{2})$

(b). $y = (\frac{3}{\sin(2x)})$

Solutions.

(a). $3 - \frac{1}{2}\csc^2(\frac{x}{2})$

(b). $-6\csc(2x)\cot(2x)$

Example 3.4.5. Find the tangent and normal lines to the curve $y = \tan\frac{\pi x}{4}$ at the point $(1, 1)$.

Solution. The slope of the tangent to $y = \tan\frac{\pi x}{4}$ at the point $(1, 1)$ is

$$\frac{dy}{dx}|_{x=1} = \frac{\pi}{4}\sec^2(\frac{\pi x}{4})|_{x=1} = \frac{\pi}{4}\sec^2(\frac{\pi}{4}) = \frac{\pi}{4}(\sqrt{2})^2 = \frac{\pi}{2}.$$

The tangent line is $y = 1 + \frac{\pi}{2}(x - 1)$ or $y = \frac{\pi}{2}x - \frac{\pi}{2} + 1$.
The normal has slope $-\frac{2}{\pi}$, so its point-slope equation is $y = 1 - \frac{2}{\pi}(x - 1)$ or $y = \frac{2}{\pi}x + \frac{2}{\pi} + 1$.

We now consider derivatives of inverse trigonometric functions.

• The function $y = \sin^{-1} x \quad (-1 < x < 1)$ has its inverse $x = \sin y \quad (-\frac{\pi}{2} < y < \frac{\pi}{2})$ are inverses of each other. Thus

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{(\sin y)'} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}.$$

• The function $y = \cos^{-1} x \quad (-1 < x < 1)$ has its inverse $x = \cos y \quad (0 < y < \pi)$ are inverses of each other. Thus

$$\frac{d}{dx}\cos^{-1} x = \frac{1}{(\cos y)'} = \frac{1}{-\sin y} = -\frac{1}{\sqrt{1 - \cos^2 y}} = -\frac{1}{\sqrt{1 - x^2}}.$$

• The function $y = \tan^{-1} x \quad (-\infty < x < \infty)$ has its inverse $x = \tan y \quad (-\frac{\pi}{2} < y < \frac{\pi}{2})$ are inverses of each other. Thus

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{(\tan y)'} = \cos^2 y = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}.$$

• The function $y = \cot^{-1} x \quad (-\infty < x < \infty)$ has its inverse $x = \cot y \quad (0 < y < \pi)$ are inverses of each other. Thus

$$\frac{d}{dx}\cot^{-1} x = \frac{1}{(\cot y)'} = -\frac{1}{1 + x^2}.$$

3.5 Derivatives of Exponential and Logarithmic Functions

The exponential$y = a^x \quad (0 < a \neq 1)$ and logarithm function $x = \log_a y \quad (0 < y < \infty)$, are inverses of each other. By the definition of the derivative/First Principles:

$$\begin{aligned} \tfrac{d}{dx}a^x &= \lim_{\Delta x \to 0} \tfrac{a^{x+\Delta x} - a^x}{\Delta x} \\ &= a^x \lim_{\Delta x \to 0} \tfrac{a^{\Delta x} - 1}{\Delta x} \\ &= a^x \ln a. \end{aligned}$$

In particular, if $a = e$,

$$\frac{d}{dx}e^x = e^x \ln e = e^x.$$

For the logarithmic function $y = \log_a x$, consider a point $x > 0$. By the definition of derivative

$$\begin{aligned} \tfrac{d}{dx}\log_a x &= \lim_{\Delta x \to 0} \tfrac{\log_a(x+\Delta x) - \log_a x}{\Delta x} \\ &= \tfrac{1}{x}\lim_{\Delta x \to 0} \tfrac{\log_a(1+\frac{\Delta x}{x})}{\frac{\Delta x}{x}} \end{aligned}$$

Since $\lim_{\Delta x \to 0} \frac{\log_a(1+t)}{t} = \log_a e$, by substituting $t = \frac{\Delta x}{x}$, it follows that

$$\frac{d}{dx}\log_a x = \frac{\log_a e}{x}................(\star)$$

Note that this result can be deduced by using

$$(\log_a y)' = \frac{1}{(a^x)'} = \frac{1}{x^x \ln a} = \frac{1}{y \ln a} = \frac{\log_a e}{y}.$$

By setting $a = e$, in $(\star)$, we find

$$\frac{d}{dx}\ln x = \frac{1}{x} \qquad (x > 0)$$

3.6 Derivatives of Hyperbolic and Inverse Hyperbolic Functions

$$\frac{d}{dx}\cosh x = \frac{d}{dx}(\frac{e^x + e^{-x}}{2}) = \frac{e^x - e^{-x}}{2} = \sinh x.$$

Similarly,

$$\frac{d}{dx}\sinh x = \cosh x.$$

$$\begin{aligned} \tfrac{d}{dx}\tanh x = \tfrac{d}{dx}(\tfrac{\sinh x}{\cosh x}) &= \tfrac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} \\ &= \tfrac{1}{\cosh^2 x} \\ &= (\text{sech x})^2 \end{aligned}$$

Likewise

$$\frac{d}{dx}\text{coth x} = -(\text{csch x})^2$$

Now, we compute the derivatives of the inverse hyperbolic functions.

$$\frac{d}{dx}\cosh^{-1} x = \frac{d}{dx}[\ln(x + \sqrt{x^2 - 1})] = \frac{\sqrt{x^2 - 1} + x}{(x + \sqrt{x^2 - 1})\sqrt{x^2 - 1}} = \frac{1}{\sqrt{x^2 - 1}}, \quad x > 1.$$

$$\frac{d}{dx}\sinh^{-1} x = \frac{d}{dx}[\ln(x + \sqrt{x^2 + 1})] = \cdots = \frac{1}{\sqrt{x^2 + 1}}$$

$$\frac{d}{dx}\tanh^{-1}x = \frac{d}{dx}[\frac{1}{2}\ln\frac{1+x}{1-x}] = \frac{1}{2}[\frac{1-x}{1+x}.\frac{2}{(1-x)^2}] = \frac{1}{1-x^2}, \quad |x|<1.$$

and

$$\frac{d}{dx}\coth^{-1}x = \frac{1}{1-x^2}, \qquad |x|>1.$$

3.7 Derivatives and Indeterminate limits of type $\frac{0}{0}$ and $\frac{\infty}{\infty}$

Indeterminate limits of type $\frac{0}{0}$ and $\frac{\infty}{\infty}$ can be evaluated using L'Hôpital's Rule.

3.7.1 L'Hôpital's First Rule for the Indeterminate form $\frac{0}{0}$

Theorem 3.7.1. *(**L'Hôpital's First Rule**)Suppose that the functions f and g are differentiable in a deleted $\sigma-$ neighborhood $(a-\sigma, a+\sigma)-\{a\}$ $(\sigma>0)$ of a point a. Moreover, suppose that*

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0,$$

and $g(x)\neq 0$ in the deleted neighborhood of a. Then if the limit $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ exists, then so does $\lim_{x\to a}\frac{f(x)}{g(x)}$ and

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

Remark. If it turns out that the quotient $\frac{f'(x)}{g'(x)}$ is again indeterminate, then L'Hôpital's Rule can be applied a second, a third, etc, time and

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)}.$$

Example 3.7.2. Compute the limit $\lim_{x\to 0}\frac{\cos x-1}{x^2}$.

Solution. The quotient has the indeterminate form $\frac{0}{0}$, and so

$$\lim_{x\to 0}\frac{\cos x-1}{x^2} = \lim_{x\to 0}\frac{-\sin x}{2x} = \lim_{x\to 0}\frac{-\cos x}{2} = \frac{-1}{2}.$$

Remark. Suppose that f and g are defined on the set $\mathbb{R}-[-\sigma,\sigma]$ and $g'(x)\neq 0$, $x\in\mathbb{R}-[-\sigma,\sigma]$ $(\sigma>0)$. Assume that the limit

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\star)$$

exists. Then the functions $g_1(t)=g(\frac{1}{t})=g(x)$ and $f_1(t)=f(\frac{1}{t})=f(x)$ (where $t=\frac{1}{x}$) are defined and differentiable on $\frac{-1}{\delta}<t<\frac{1}{\delta}$ $\quad (t\neq 0)$.
Besides, the derivative $g_1'(t)\neq 0$ in the deleted $\frac{1}{\delta}$-neighborhood of the point $t=0$ takes the form

$$g_1'(t) = g'(\frac{1}{t})(-\frac{1}{t^2}) = g'(x)(-x^2)\neq 0.$$

On the other hand, the existence of the limit ($\star$) implies the following limit exists

$$\lim_{t\to 0}\frac{f_1'(t)}{g_1'(t)}=\lim_{t\to 0}\frac{f_1'(\frac{1}{t})}{g_1'(\frac{1}{t})}=\lim_{x\to\infty}\frac{f'(x)}{g'(x)}......(\star\star)$$

Therefore by the theorem, the limit

$$\lim_{t\to 0}\frac{f_1(t)}{g_1(t)}=\lim_{x\to\infty}\frac{f(x)}{g(x)}$$

exists and

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{t\to 0}\frac{f_1(t)}{g_1(t)}=\lim_{t\to 0}\frac{f_1'(t)}{g_1'(t)}=\lim_{x\to\infty}\frac{f'(x)}{g'(x)}.$$

Example 3.7.3. Find $\lim_{x\to 0}\frac{x^2}{\ln(1+x^2)}$

Solution. This is an indeterminate form $\frac{0}{0}$. By the theorem we have

$$\lim_{x\to 0^+}\frac{x^2}{\ln(1+x^2)}=\lim_{x\to 0^+}\frac{2x}{\frac{1}{1+x^2}}=\lim_{x\to 0^+}2x(1+x^2)=0.$$

Example 3.7.4. Find $\lim_{x\to\infty}x\ln(1+\frac{1}{x})$.

Solution. Obviously,

$$\lim_{x\to\infty}x\ln(1+\frac{1}{x})=\lim_{x\to\infty}\frac{\ln(1+\frac{1}{x})}{\frac{1}{x}}=\lim_{x\to\infty}\frac{-\frac{1}{x(x+1)}}{-\frac{1}{x^2}}=\lim_{x\to\infty}\frac{x}{x+1}=1.$$

3.7.2 L'Hôpital's Second Rule for the Indeterminate form $\frac{\infty}{\infty}$

Recall that if two functions f and g are defined in a neighborhood of a point$x=a$, such that $\lim_{x\to a}f(x)=\infty$, $\lim_{x\to a}g(x)=\infty$, then we say that the quotient $\frac{f(x)}{g(x)}$ has the indeterminate form $\frac{\infty}{\infty}$ at $x=a$.

Theorem 3.7.5. ***(L'Hôpital's Second Rule)****Suppose that the functions f and g are differentiable in a deleted $\sigma-$ neighborhood $(a-\sigma,a+\sigma)-\{a\}$ $(\sigma>0)$ of a point a. Moreover, suppose that*

$$\lim_{x\to a}f(x)=\lim_{x\to a}g(x)=\infty,$$

and $g'(x)\neq 0$ in the deleted neighborhood of a. Then if the limit $\lim_{x\to a}\frac{f'(x)}{g'(x)}$.......($\star\star$) exists, then so does $\lim_{x\to a}\frac{f(x)}{g(x)}$ and

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

Remark. If the limit ($\star\star$) becomes ∞, then so $\lim_{x\to a}\frac{g'(x)}{f'(x)}=0$, it follows that $\lim_{x\to a}\frac{g(x)}{f(x)}=0$. Hence, we have

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\infty.$$

Example 3.7.6. Find $\lim_{x\to 0^+}\sqrt{x}\ln x$.

Solution. $\lim_{x\to 0^+}\sqrt{x}\ln x = \lim_{x\to 0^+}\frac{\ln x}{x^{-\frac{1}{2}}} = \lim_{x\to 0^+}\frac{\frac{1}{x}}{(-\frac{1}{2})x^{-\frac{3}{2}}} = -2\lim_{x\to 0^+}\sqrt{x} = 0.$

Example 3.7.7. Find $\lim_{x\to\infty}\frac{\ln x}{x}$.

Solution. $\lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = \lim_{x\to\infty}\frac{1}{x} = 0.$

Example 3.7.8. Find $\lim_{x\to\infty}\frac{x^n}{e^x}$.

Solution. $\lim_{x\to\infty}\frac{x^n}{e^x} = \lim_{x\to\infty}\frac{nx^{n-1}}{e^x} = \lim_{x\to\infty}\frac{nx^{n-2}}{e^x} = \cdots = \lim_{x\to\infty}\frac{n!}{e^x} = 0.$

3.7.3 Indeterminate forms $0.\infty, \infty-\infty, 1^\infty, 0^0, 0^\infty, \infty^0$

The indeterminate forms $0.\infty, \infty-\infty, 1^\infty, 0^0, \infty^0$ can be converted by algebraic substitutions and tricks into a from of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and L'Hôpital's Rule used to compute them.
Suppose that we try to find the limit of $y = f(x)^{g(x)}$ where $g(x > 0)$, when the limits of f and g as $x \to a$ are such that one of the indeterminate forms $1^\infty, 0^0, 0^\infty, \infty^0$ is produced. First, we calculate the logarithm:

$$\ln y = g(x)\ln f(x).$$

For each of these three cases, $\ln y = g(x)\ln f(x)$ has the form $0.\infty$ as $x \to a$. Thus, it is easy to see that if

$$z = p(x)q(x), ...(\star)$$

where $\lim_{x\to a} p(x) = 0, \lim_{x\to a} q(x) = \infty$,
the indeterminate form $0.\infty$ can be converted by algebraic substitutions into either the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.
Expressing $(\star)$ in the form

$$z = \frac{p(x)}{\frac{1}{q(x)}}.......(\star\star)$$

or

$$z = \frac{q(x)}{\frac{1}{p(x)}}.......(\star\star\star)$$

we see that $(\star\star)$ and $(\star\star\star)$ have the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$, respectively.

Example 3.7.9. Find $\lim_{x\to 0} x^x$.

Solution. This is of indeterminate form 0^0. Let $y = x^x$. Then $\ln y = x\ln x = \frac{\ln x}{\frac{1}{x}}$, and by passing to the limit as $x \to 0$, we have

$$\lim_{x\to 0} x\ln x = \lim_{x\to 0}\frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = -\lim_{x\to 0} x = 0.$$

Therefore $\ln\lim_{x\to 0} y = 0$, and so $\lim_{x\to 0} x^x = e^0 = 1$.

Example 3.7.10. Find $\lim_{x\to 0}(1+x^2)^{\frac{1}{x^2}}$.

Solution. This is of indeterminate form 1^∞. Let $y=(1+x^2)^{\frac{1}{x^2}}$. Then $\ln y = \frac{1}{x^2}\ln(1+x^2)$, and by passing to the limit as $x\to 0$ we have

$$\lim_{x\to 0}\frac{1}{x^2}\ln(1+x^2) = \lim_{x\to 0}\frac{\ln(1+x^2)}{x^2} = \lim_{x\to 0}\frac{(\frac{1}{1+x^2})(2x)}{2x} = \lim_{x\to 0}\frac{1}{1+x^2} = 1.$$

Therefore $\ln\lim_{x\to 0} y = 1$, and so $\lim_{x\to 0}(1+x^2)^{\frac{1}{x^2}} = e$.

Exercises. 1. Use L'Hôpital's Rule to find the limits

(a). $\lim_{x\to 0}\frac{\tan x - x}{x-\sin x}$, `Ans.` 2

(b). $\lim_{x\to 0}\frac{\sin ax}{\sin bx}$, `Ans.` $\frac{a}{b}$.

(c). $\lim_{x\to\frac{\pi}{4}}\frac{\sqrt[3]{\tan x}-1}{2\sin^2 x - 1}$, `Ans.` $\frac{1}{3}$.

(d). $\lim_{x\to\frac{\pi}{4}}(\tan x)^{\tan 2x}$, `Ans.` $\frac{1}{e}$.

(e). $\lim_{x\to 0}\frac{\ln(\cos ax)}{\ln(\cos bx)}$, `Ans.` $(\frac{a}{b})^2$.

(f). $\lim_{x\to a}\frac{a^x - x^a}{x-a}$ $(a>0)$, `Ans.` $a^a(\ln a - 1)$.

3.8 Differentiation of Implicit and Parametric Functions

In this section we consider functions defined implicitly and functions defined parametrically.

3.8.1 Implicit Differentiation

Sometimes we are not given y as a function of x explicitly, but instead have an equation connecting them which we may be unable to solve explicitly for either x or y. We may still want to find $\frac{dy}{dx}$.

Example 3.8.1. Find $\frac{dy}{dx}$ if $y^2 = x$.

Solution. We differentiate both sides of the equation with respect to x, treating y as a differentiable but otherwise unknown function of x. Therefore

$$\begin{aligned} y^2 &= x \\ 2y\frac{dy}{dx} &= 1 \ (Chain\ \ Rule) \\ \frac{dy}{dx} &= \frac{1}{2y} \end{aligned}$$

Example 3.8.2. Find the gradient $\frac{dy}{dx}$ at the point (1,2) on the curve whose equation is $x^3 - 5xy^2 + y^3 + 11 = 0$.

Solution. We can not find $y(x)$ explicitly in terms of x. We therefore have to use the Chain Rule to differentiate the y^2 and y^3 terms and the product rule for the second term involving x and y. Differentiating with respect to x, we have

$$3x^2 - 5(x.2y\frac{dy}{dx} + y^2) + 3y^2\frac{dy}{dx} = 0$$

Rearranging this gives

$$(3y^2 - 10xy)\frac{dy}{dx} = 5y^2 - 3x^2,$$

and therefore

$$\frac{dy}{dx} = \frac{5y^2 - 3x^2}{3y^2 - 10xy}.$$

The gradient at the point (1,2) is then found by substituting these values for x and y the expression, giving $\frac{-17}{8}$.

Example 3.8.3. Find $\frac{d^2y}{dx^2}$ if $2x^3 - 3y^2 = 7$.

Solution. We differentiate both sides with respect to x to find $y' = \frac{dy}{dx}$:

$$\begin{array}{ccc} 2x^3 - 3y^2 & = & 7 \\ \frac{d}{dx}(2x^3) - \frac{d}{dx}(3y^2) & = & \frac{d}{dx}(7) \\ 6x^2 - 6yy' & = & 0 \\ x^2 - yy' & = & 0 \\ y' & = & \frac{x^2}{y} \quad (when\ y \neq 0). \end{array}$$

We now apply the Quotient Rule to find y'':

$$y'' = \frac{d}{dx}(\frac{x^2}{y}) = \frac{2xy - x^2y'}{y^2} = \frac{2x}{y} - \frac{x^2}{y^2}y'.$$

Finally, we substitute $y' = \frac{x^2}{y}$ to express y'' in terms of x and y:

$$y'' = \frac{2x}{y} - \frac{x^2}{y^2}(\frac{x^2}{y}) = \frac{2x}{y} - x^4y^3.$$

3.8.2 Logarithmic Differentiation

This is an application of implicit differentiation.

Example 3.8.4. Differentiate $y = x^{\sin x}$.

Solution. We take logarithms of both sides of the equation to give

$$\ln y = \ln(x^{\sin x}) = \sin x \ln x.$$

We deal with the LHS using implicit differentiation, and the RHS using the product rule. This gives

$$\frac{1}{y}\frac{dy}{dx} = \cos x.\ln x + \sin x.(\frac{1}{x}).$$

Therefore

$$\frac{dy}{dx} = y(\cos x.\ln x + \sin x).$$

3.8.3 Parametric Differentiation

Equations of curves are often given parametrically. For example the ellipse is sometimes specified by

$$x = a\cos t, y = b\sin t, 0 \leq t \leq 2\pi.$$

We want to find $\frac{dy}{dx}$(the gradient), but the parametric equations can only be differentiated with respect to t. We can approach this in two ways.
Firstly, we can use the Chain Rule to give

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt},$$

so

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}},$$

provided $\frac{dx}{dt} \neq 0$.

Example 3.8.5. Find the gradient at an arbitrary point t on the ellipse specified by $x = a\cos t, y = b\sin t, 0 \leq t \leq 2\pi$.

Solution. The gradient is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{b\cos t}{-a\sin t} = -\frac{b}{a}\cot t.$$

Note that here $\sin t \neq 0$, and thus t excludes the points $0, \pm\pi, \pm 2\pi, ...,$ where the tangent to the ellipse is parallel to the y-axis.

Example 3.8.6. Find $\frac{dy}{dx}$ given that $x = t^2, y = t^3$.

Solution. These are parametric equations of a curve known as a semicubical parabola. The derivative is given

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2}{2t} = \frac{3t}{2} \quad (t \neq 0).$$

We can also eliminate the parameter t to give $y^3 = x^3$, and so we could also find the derivative using implicit differentiation, as follows

$$2y\frac{dy}{dx} = 3x^2,$$

and so, $\frac{dy}{dx} = \frac{3x^2}{2y}, \quad (y \neq 0)$.

Example 3.8.7. Given $x = a\cos t, y = b\sin t, 0 \le t \le 2\pi$, find $\frac{d^2y}{dx^2}$.

Solution. It is possible to find a general formula for the second derivative. We write

$$\frac{d^2y}{dx^2} = \frac{dY}{dx},$$

where $Y = \frac{dy}{dx}$. Applying parametric differentiation formula to Y gives

$$\frac{dY}{dx} = \frac{\frac{dY}{dt}}{\frac{dx}{dt}}.$$

Thus

$$\frac{d^2y}{dx^2} = \frac{dY}{dt} = \frac{\frac{dY}{dt}}{\frac{dx}{dt}} = \frac{\frac{-b}{a}\csc^2 t}{-a\sin t} = -\frac{b}{a^2\sin^3 t},$$

provided that $\sin t \neq 0$.

Caution. A common mistake is to try to find $\frac{d^2y}{dx^2}$ by differentiating the formula obtained for $\frac{dy}{dx}$ with respect to x. This `WRONG`!

3.8.4 Differentiating Inverse Functions using Implicit Differentiation

Example 3.8.8. Find the derivative of $y = \sinh^{-1} x$.

Solution. Suppose that $y = \sinh^{-1} x$, so that $x = \sinh y$. Differentiating with respect to x gives

$$1 = \cosh y \frac{dy}{dx},$$

so that

$$\frac{dy}{dx} = \frac{1}{\cosh y}.$$

We want the answer in terms of x, so we have to find $\cosh y$ in terms of $x = \sinh y$. Using the hyperbolic identity

$$\cosh^2 y - \sinh^2 y = 1,$$

gives

$$\cosh y = \sqrt{1 + \sinh^2 y},$$

where we use the positive square root because $\cosh y$ is always positive. Therefore

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}}.$$

Exercise. Use implicit differentiation to verify the derivatives of inverse trigonometric and hyperbolic functions proved earlier.

3.9 Linear Approximations and Differentials

Sometimes we can approximate complicated functions with simpler ones that give the accuracy we want for specific applications.

3.9.1 Differentials

The Newton quotient $\frac{\Delta y}{\Delta x}$ is a quotient of two quantities Δy and Δx. If y is a continuous function of x, then $\Delta y \to 0$ as $\Delta x \to 0$, so $\frac{dy}{dx}$ is the meaningless quantity $\frac{0}{0}$. We refer to dx as the **differential of** x and dy as the **differential of** y as a function of x and dx:

$$dy = \frac{dy}{dx}dx = f'(x)dx.$$

For example, if $y = x^2$, then $dy = 2xdx$.

Remark. The differentials dy and dx were originally regarded (by Leibniz and his successors as "`infinitesimals`"(meaning *infinitely small but nonzero*)). If one quantity, say y, is a function of another quantity x, that is, $y = f(x)$, we sometimes want to know how a change in the value of x by an amount Δx will affect the value of y. The exact change Δx in y is given by

$$\Delta y = f(x + \Delta x) - f(x),$$

but if the change Δx is small, then we can get a good approximation to Δy by using the fact that $\frac{\Delta y}{\Delta x}$ is approximately the derivative $\frac{dy}{dx}$. Thus

$$\Delta y = \frac{\Delta y}{\Delta x}\Delta x \approx \frac{dy}{dx}\Delta x = f'(x)\Delta x.$$

It is often convenient to represent this approximation in terms of differentials; if we denote the change in x by dx instead of Δx, then the change Δy in y is approximated by the differential dy, that is

$$\Delta y \approx dy = f'(x)dx.$$

Example 3.9.1. Without using a calculator, determine by approximately how much the value of $\sin x$ increases as x increases from $\frac{\pi}{3}$ to $\frac{\pi}{3} + 0.006$. To 3 decimal places, what is the value of $\sin(\frac{\pi}{3} + 0.006)$?

Solution. If $y = \sin x, x = \frac{\pi}{3} \approx 1.0472$, and $dx = 0.006$, then

$$dy = \cos x dx = \cos(\frac{\pi}{3})dx = \frac{1}{2}(0.006) = 0.003.$$

Thus the change in the value of $\sin x$ is approximately 0.003, and

$$\sin(\frac{\pi}{3} + 0.006) \approx \sin(\frac{\pi}{3}) + 0.003 = \frac{\sqrt{3}}{2} + 0.003 = 0.869,$$

rounded to 3 decimal places.

Sometimes changes in a quantity are measured with respect to the size of the quantity. The relative change in x is the ratio $\frac{dx}{x}$ if x changes by amount dx. The percentage change in x is the relative change expressed as a percentage:

$$\textit{relative change in } x = \frac{dx}{x}$$

$$\textit{percentage relative change in } x = 100\frac{dx}{x}$$

Example 3.9.2. By approximately what percentage does the area of a circle increase if the radius increases by 2%?

Solution. The area A os a circle is given in terms of radius r by $A = \pi r^2$. Thus

$$\Delta A \approx dA = \frac{dA}{dr}dr = 2\pi r dr.$$

We divide this approximation by $A = \pi r^2$ to get an approximation that links the relative changes in A and r:

$$\frac{\Delta A}{A} \approx \frac{dA}{A} = \frac{2\pi r dr}{\pi r^2} = 2\frac{dr}{r}.$$

If r increases by 2%, then $dr = \frac{2}{100}r$, so

$$\frac{\Delta A}{A} \approx 2\frac{2}{100} = \frac{4}{100}.$$

Thus, A increases by approximately 4%.

3.9.2 Linearizations are Linear Replacement Formulas

Remark. The tangent to a curve $y = f(x)$ lies close to the curve near the point of tangency. For a small interval to either side, the y-values along the tangent line give good approximations to the y-values on the curve. Therefore, to simplify the expression for the function near this point, we propose to replace the formula for f over this interval by the formula for its tangent line. If the tangent passes through the point $P(a, f(a))$ with slope $f'(a)$, then its point-slope equation is

$$y - f(a) = f'(a)(x - a)$$

or

$$y = f(a) + f'(a)(x - a).$$

, the tangent line is the graph of the function

$$L(x) = f(a) + f'(a)(x - a)..........(\star)$$

For as long as the line remains close to the graph of f, $L(x)$ will give a good approximation to $f(x)$.

Example 3.9.3. Find the linearization of $f(x) = \sqrt{1+x}$ at $x = 0$.

Solution. We evaluate ($\star$) for $f(x) = \sqrt{1+x}$ and $a = 0$. The derivative of $f(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}} = \frac{1}{2\sqrt{1+x}}$. Its value at $x = 0$ is $\frac{1}{2}$. We substitute this along with $a = 0$ and $f(0) = 1$ into ($\star$):

$$L(x) = f(a) + f'(a)(x-a) = 1 + \frac{1}{2}(x-0) = 1 + \frac{x}{2}.$$

Example 3.9.4. Find the linearization of $f(x) = \tan x$.

Solution. $f(x) = \tan x$, $a = 0$. Since $f(0) = 0, f'(0) = \sec^(0) = 1$, we have $L(x) = 0 + 1(x-0) = x$. Near $x = 0$

$$\tan x \approx x.$$

Example 3.9.5. Use linear approximation formula to estimate $\sqrt{1.1}$.

Solution. Let $f(x) = \sqrt{x}$. Then we must compute $f(1.1)$. By definition of derivative,

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})} = \frac{1}{2\sqrt{x}}.$$

Setting $x = 1, x + \Delta x = 1.1$, it follows that

$$f(1.1) \approx f(1) + f'(1)(0.1)$$

or

$$\sqrt{1.1} \approx \sqrt{1} + \frac{1}{2\sqrt{1}}(0.1) = 1.05.$$

Consequently, $\sqrt{1.1} \approx 1.05$.

Example 3.9.6. Use linear approximation formula to estimate $\sin 46°$.

Solution. Let $f(x) = \sin x$. Then we must compute $f(46)$. Clearly, $f'(x) = \cos x$ and by linear approximation

$$\sin(x + \Delta x) \approx \sin x + \cos x \Delta x.$$

We estimate the value of $\sin 46°$. Since $x + \Delta x = \frac{\pi}{4} + \frac{\pi}{180}$ ($1° = \frac{\pi}{180}$ *radians*), and $x = \frac{\pi}{4}$, we have

$$\sin 46° = \sin(\frac{\pi}{4} + \frac{\pi}{180}) \approx \sin\frac{\pi}{4} + \frac{\pi}{180}\cos\frac{\pi}{4},$$

or

$$\sin 46° \approx \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\frac{\pi}{180} = 0.7071 + 0.7071(0.0175) = 0.7194.$$

3.9.3 Average and Instantaneous Rates of Change

Definition 3.9.7. The **average rate of change** of a function $f(x)$ with respect to x over an interval from a to $a+h$ is

$$\frac{f(a+h) - f(a)}{h}.$$

The (instantaneous) rate of change of f with respect to x at $x = a$ is the derivative

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists.

Example 3.9.8. How fast is area A of a circle increasing with respect to its radius when the radius is 5 m?

Solution. The rate of change of the area with respect to the radius is

$$\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r.$$

When $r = 5\ m$, the area is changing at the rate of $2\pi(5) = 10\pi\ m^2/min$. This means that a small change $\Delta r\ m$ in the radius when the radius is 5 m would result in a change of about $10\pi\ m^2$ in the area of the circle.

3.10 Exercises

1.(a). Use First Principles to find $\frac{dy}{dx}$ for each of the following functions.

(i). $y = f(x) = x^2$.

(ii). $y = \cos(2x)$.

(iii). $y = \frac{1}{2x}$.

(iv). $y = \frac{2-x}{2+x}$.

(v). $y = e^{\alpha x}$.

(vi). $y = \log_a x$.

Solution(vi). Let $y = \log_a x$. Then $y + \triangle y = \log_a(a + \triangle x)$. Thus

$$\triangle y = \log_a(x + \triangle x) - \log_a x = \log_a \frac{x + \triangle x}{x} = \log_a(1 + \frac{\triangle x}{x}).$$

Therefore

$$\frac{\triangle y}{\triangle x} = \frac{1}{\triangle x}\log_a(1 + \frac{\triangle x}{x}) = \frac{1}{x}\frac{x}{\triangle x}\log_a(1 + \frac{\triangle x}{x}) = \frac{1}{x}\log_a(1 + \frac{\triangle x}{x})^{\frac{x}{\triangle x}}$$

and

$$\frac{dy}{dx} = \frac{1}{x}\lim_{\triangle x \to 0}\log_a(1 + \frac{\triangle x}{x})^{\frac{x}{\triangle x}} = \frac{1}{x}\log_a\left[\lim_{\triangle x \to 0}(1 + \frac{\triangle x}{x})^{\frac{x}{\triangle x}})\right] = \frac{1}{x}\log_a e.$$

When $a = e$, $\log_a e = \log_e e = 1$ and $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

(b). Find the derivatives of

(i). $y = \frac{1-x^2}{1+x^2}$.

(ii). $y = (2 + \frac{3}{t})^{-10}$.

(iii). $y = \sqrt{x\sqrt{x}}$.

(iv). $y = x^{\sin(x)}$.

(v). $y = \sec(2x^3 + 1)$.

(vi). $y = a^x$.

Solution. Let $y = a^x$. Then $\ln y = x \ln a$ and $\frac{d}{dx}(\ln y) = \frac{1}{y}\frac{dy}{dx} = \ln a$. Thus $\frac{dy}{dx} = y \ln a = a^x \ln a$. When $a = e$, $\ln a = \ln e = 1$ and we have $\frac{d}{dx}(e^x) = e^x$.

(c). Show that the derivative of $y = f(x) = |x|$ does not exist at $x = 0$.

Solution. This function is continuous at all values of x. For $x < 0$, $f(x) = -x$ and $f'(x) = \lim_{h\to 0} -\frac{(x+h)-(-x)}{h} = -1$; for $x > 0$, $f(x) = x$ and $f'(x) = 1$.
At $x = 0$, $f(x) = 0$ and $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{|h|}{h}$. As $h \to 0^-$, $\frac{|h|}{h} \to -1$; but as $h \to 0^+$, $\frac{|h|}{h} \to 1$.
Hence the derivative does not exist.

2.(a). Find $\frac{dy}{dx}$ in each case.

(i). $x^3 - xy + y^3 = 1$.

(ii). $x^2y + xy^2 = 6$.

(iii). $y^3 + y = 2\cos x$ at the point $(0, 1)$.

(iv). $y = \ln(x^2 + 3x + 1)$.

(v). $y = \ln(\frac{x^2+1}{x^3-x})$.

(b). Find the tangent line to the curve

(i). $x^3 + y^2 = 2$ at the point $(1, 1)$. `Solution`: $y = -\frac{3}{2}x + \frac{5}{2}$

(ii). $x^{\frac{3}{2}} + 2y^{\frac{3}{2}} = 17$. `Solution`: $y = -\frac{1}{4} + \frac{17}{4}$

(iii). $x^3y^3 + y^2 = x + y$ at the point $(1, 1)$.

(iv). $x + \sqrt{xy} = 6$ at the point $(4, 1)$.

(c).(i). Given that $xy + x - 2y - 1 = 0$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(ii). If $y = \ln(x + \sqrt{1+x^2})$, show that $y' = \frac{1}{\sqrt{1+x^2}}$.

3.(a). If $x = \cos(2t-1)$ and $y = \sin(3t+1)$, find $\frac{dy}{dx}$.

(b). Find $\frac{dy}{dx}$ if $y = x^{x^x}$.

4. Suppose that the temperature at a certain location t hours after noon on a certain day is T°C, where $T = \frac{1}{3}t^3 - 3t^2 + 8t + 10 \quad (\text{for } 0 \leq t \leq 5)$. How fast is the temperature rising or falling at 1 : 00 pm? at 3 : 00 pm? At what instants is the temperature stationary?

Solution. The rate of change of the temperature is $\frac{dT}{dt} = t^2 - 6t + 8 = (t-2)(t-4)$.

- If $t = 1$, then $\frac{dT}{dt} = 3$, so the temperature is *rising* at the rate of 3 degrees C/h at 1 : 00pm.
- If $t = 3$, then $\frac{dT}{dt} = -1$, so the temperature is *falling* at the rate of 1 degree C/h at 3 : 00pm.
- The temperature is stationary when $\frac{dT}{dt} = 0$, that is, at 2 : 00 pm and 4 : 00 pm.

5.(a). If $z = \tan\frac{x}{2}$, then show that $\frac{dx}{dz} = \frac{2}{1+z^2}, \quad \sin x = \frac{2z}{1+z^2}$ and $\cos x = \frac{1-z^2}{1+z^2}$.

(b). Find the linearization of the given function at the given point.

(i). $y = x^2$ about $x = 3$.

(ii). $y = \sqrt{4-x}$ about $x = 0$.

(iii). $y = \sin x$ about $x = \pi$.

(c).(i). By approximately how much does the area of a square increase if its side length increases from 10 cm to 10.4 cm?

(ii). By about how much must the length of a cube decrease from 20 cm to reduce the volume of the cube by 12 cm^3?

Chapter 4

ANTI-DERIVATIVES OF FUNCTIONS AND APPLICATIONS TO AREAS

Throughout Chapter 3 we have been concerned with the problem of finding the derivative f' of a given function f. The reverse problem-given the derivative f', find f-is also interesting and important. It is the problem studied in *integral calculus* and is generally more difficult to solve than the problem of finding a derivative.

4.1 Anti-Derivatives

We begin by defining an anti derivative of a function f to be a function F whose derivative is f. It is appropriate to require that $F'(x) = f(x)$ on an interval.

Definition 4.1.1. An **antiderivative** of a function f on an interval I is another function F satisfying $F'(x) = f(x)$ for x in I. The reverse process of determining F from f is called **anti-differentiation** or **integration**.

Remark. Antiderivatives are not unique; indeed, if C is any constant, then $F(x) = x+$ is an antiderivative of $f(x) = 1$ on any interval. You can always add a constant to an antiderivative F of a function f on an interval and get another antiderivative of f. More importantly, all antiderivatives of f on an interval can be obtained by adding constants to any particular one. If F and G are both antiderivatives of I on an interval I, then

$$\frac{d}{dx}(G(x) + F(x)) = f(x) - f(x) = 0,$$

on I, so $G(x) - F(x) = C$ (a constant) on I. Thus $G(x) = F(x) + C$ on I.

4.1.1 Indefinite Integral of a Function

Example 4.1.2. If $f(x)$ is a derivative, the set of all antiderivatives of f is called the **indefinite integral** of f, denoted by the symbols

$$\int f(x)dx.$$

The sign $\int$ is called an **integral sign**. The function f is the **integrand** of the integral and x is the **variable of integration**.

Remark. We have already seen that all the anti-derivatives F of f differ by a constant. We indicate this by writing

$$\int f(x)dx = F(x) + C. \qquad (\star)$$

The constant C is called the **constant of integration**. Equation $(\star)$ is read as "*The indefinite integral of f with respect to x is $F(x) + C$*". When we find $F(x) + C$, we say that we have **evaluated** the indefinite integral.

Example 4.1.3.

Function $f(x)$	*antiderivative*	*Reversed derivative formula*
$\cos x$	$\sin x + C$	$\frac{d}{dx}\sin x = \cos x$
$\sin x$	$-\cos x + C$	$\frac{d}{dx}(-\cos x) = \sin x$
$3x^2$	$x^3 + C$	$\frac{d}{dx}x^3 = 3x^2$
$\frac{1}{2\sqrt{x}}$	$\sqrt{x} + C$	$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$
$\frac{1}{x^2}$	$-\frac{1}{x} + C$	$\frac{d}{dx}(-\frac{1}{x}) = \frac{1}{x^2}$

Example 4.1.4. Evaluate $\int(x^2 - 2x + 5)dx$

Solution.

$$\int (x^2 - 2x + 5)dx = \overbrace{\frac{x^3}{3} - x^2 + 5x}^{An\ antiderivative\ of\ f(x)=x^2-2x+5} + C.$$

Remark. To evaluate $\int f(x)dx$, find an antiderivative $F(x)$ of $f(x)$ and then add a constant C.

4.1.2 Integration Formulas

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$ (Power Rule)
2. $\int \sin kx dx = -\frac{\cos kx}{k} + C$ (Chain Rule)
3. $\int \cos kx dx = \frac{\sin kx}{k} + C$ (Chain Rule)
4. $\int \sec^2 x dx = \tan x + C$
5. $\int \sec x \tan x dx = \sec x + C$
6. $\int \csc^x dx = -\cot x + C$
7. $\int \csc x \cot x dx = -cscx + C$

Example 4.1.5. a). $\int x^5 dx = \frac{x^6}{6} + C$

b). $\int \sin 2x dx = -\frac{\cos 2x}{2} + C$

c). $\int \cos \frac{x}{2} dx = \int \cos \frac{1}{2}x = \frac{\sin \frac{1}{2}x}{\frac{1}{2}}$+C=2sin $\frac{x}{2} + C$

Example 4.1.6. $\int x \cos x dx = x \sin x + \cos x + C$

Reason: The derivative of the right-hand side is the integrand:

$$\frac{d}{dx}(x \sin x + \cos x + C) = x \cos x + \sin x - \sin x + 0 = x \cos x.$$

4.1.3 Rules for Indefinite Integrals

1. $\int \frac{dF}{dx} dx = F(x) + C$
2. $\frac{d}{dx} \int f(x) dx = f(x)$
3. $\int k f(x) dx = k \int f(x) dx$ (k a constant)
4. $\int -f(x) dx = -\int f(x)$
5. $\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx$

4.2 Techniques of Integration

This section is completely concerned with how to evaluate integrals.

4.2.1 Integration by Substitution

A change of variable can often turn an unfamiliar integral into one we can evaluate. The method of doing this is called the **Substitution Method.**

Example 4.2.1. Evaluate $\int (x+2)^5 dx$

Solution. Let $u = x + 2$. Then $du = dx$. Therefore

$$\int (x+2)^5 dx = \int u^5 du = \frac{u^6}{6} + C = \frac{(x+2)^6}{6} + C.$$

Example 4.2.2. Evaluate $\int \sqrt{4x-1} dx$

Solution. Let $u = 4x - 1$. Then $du = 4dx$ or $dx = \frac{1}{4} du$. Thus

$$\int \sqrt{4x-1} dx = \frac{1}{4} \int u^{\frac{1}{2}} du = \frac{1}{4} \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{1}{6} u^{\frac{3}{2}} + C = \frac{1}{6}(4x-1)^{\frac{3}{2}} + C.$$

Example 4.2.3. Evaluate $\int \cos(7x+5)dx$

Solution. Let $u = 7x+5$. Then $du = 7dx$ or $dx = \frac{1}{7}du$. Thus

$$\int \cos(7x+5)dx = \frac{1}{7}\cos u du = \frac{1}{7}\sin u + C = \frac{1}{7}\sin(7x+5) + C.$$

Example 4.2.4. Evaluate $\int x^2 \sin(x^3)dx$.

Solution. Let $u = x^3$. Then $du = 3x^2dx$ or $x^2dx = \frac{1}{3}du$. Thus

$$\int x^2\sin(x^3)dx = \int \sin u.\frac{1}{3}du = \frac{1}{3}\int \sin u du = \frac{1}{3}(-\cos u) + C = -\frac{1}{3}\cos(x^3) + C.$$

Example 4.2.5. Evaluate $\int \frac{1}{\cos^2 2x}dx$.

Solution. Note that $\sec 2x = \frac{1}{2x}$. Let $u = 2x$. Then $du = 2dx$ or $dx = \frac{1}{2}du$. Thus

$$\int \frac{1}{\cos^2 2x}dx = \int \sec^2 2x dx = \int \sec^2 u.\frac{1}{2}du = \frac{1}{2}\int \sec^2 u du = \frac{1}{2}\tan u + C = \frac{1}{2}\tan 2x + C.$$

Example 4.2.6. Evaluate $\int \sin^4 x \cos x dx$.

Solution. Let $u = \sin x$. Then $du = \cos x dx$ or $\cos x dx = du$. Thus

$$\int \sin^4 x \cos x dx = \int u^4 du = \frac{u^5}{5} + C = \frac{\sin^5 x}{5} + C.$$

4.2.2 Integration by Parts

Our next general method for anti-differentiation is called ***Integration by Parts***. Just as the method of substitution can be regarded as inverse to the Chain Rule for differentiation, so the method for integration by parts is inverse to the Product Rule for differentiation. Suppose that $U(x)$ and $V(x)$ are two differentiable functions. According to the Product Rule,

$$\frac{d}{dx}\Big(U(x).V(x)\Big) = U(x)\frac{dV}{dx} + V(x)\frac{dU}{dx}.$$

Integrating both sides of this equation and transposing terms yields

$$\int U(x)\frac{dV}{dx}dx = U(x)V(x) - \int V(x)\frac{dU}{dx}dx,$$

or more simply,

$$\int UdV = UV - \int VdU.$$

In each application of the method, we break up the given integrand into a product of two pieces, U and V', where V' is readily integrated and where $\int VU'dx$ is usually (but not always) a *simpler* integral than $\int UV'dx$. The technique is called integration by parts because it replaces one integral with the sum of an integrated term and another integral that remains to be evaluated. That is, it accomplishes only *part* of the original integration.

Example 4.2.7. Evaluate $\int xe^x dx$.

Solution. Let $U = x, dv = e^x dx$. Then $dU = dx$ and $V = e^x$. Thus

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C.$$

Note: In general, do not include a constant of integration with V or on the right-hand side until the last integral has been evaluated.
Choices should be made for U and dV in various situations. An improper choice can result in making an integral more difficult rather than easier. Look for a factor of the integrand that is easily integrated, and include dx with that factor to make up dV. Then V is the remaining factor of the integrand. Sometimes it is necessary to take $dV = dx$ only. When breaking up an integrand using integration by parts, choose U and dV so that, if possible, VdU is "*simpler*" (easier to integrate) than UdV .

Example 4.2.8. Use integration by parts to evaluate:
a). $\int \ln x dx$ b). $\int x^2 \sin x dx$ c). $\int x \tan^{-1} x dx$ d). $\int \sin^{-1} x dx$

Solution.
a). Let $U = \ln x, dV = dx$. Then $dU = \frac{dx}{x}, V = x$. Thus

$$\int \ln x dx = x \ln x - \int x \frac{1}{x} dx = x \ln x - x + C.$$

b). We apply integration by parts method twice: let $U = x^2, dV = \sin x dx$. Then $dU = 2xdx, V = -\cos x$. Thus

$$\int x^2 \sin x dx = -x^2 \cos x + 2 \int x \cos x dx.$$

Now, let $U = x, dV = \cos x dx$. Then $dU = dx, V = \sin x$. Thus

$$\int x^2 \sin x dx = -x^2 \cos x + 2 \int x \cos x dx = -x^2 \cos x + 2(x \sin x - \int \sin x dx)$$

$$= -x^2 \cos x + 2x \sin x + 2 \cos x + C.$$

c). Let $U = \tan^{-1} x, dV = xdx$. Then $dU = \frac{dx}{1+x^2}, V = \frac{1}{2}x^2$. Thus

$$\int x \tan^{-1} x dx = \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx = \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2} \int (1 - \frac{1}{1+x^2}) dx$$

$$= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}x + \frac{1}{2} \tan^{-1} x + C.$$

d). Let $U = \sin^{-1} x, dV = dx$. Then $dU = \frac{dx}{\sqrt{1-x^2}}, V = x$. Thus

$$\int \sin^{-1} x dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx.$$

Now we use integration by substitution. Let $u = 1 - x^2$. Then $du = -2xdx$ Thus

$$\int \sin^{-1} x dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx = x \sin^{-1} x + \frac{1}{2} \int u^{-\frac{1}{2}} du = x \sin^{-1} x + u^{\frac{1}{2}} + C$$

$$= x\sin^{-1} x + \sqrt{1-x^2} + C.$$

Remark. The following are two useful rules of thumb for choosing U and dV:

(i) If the integrand involves a polynomial multiplied by an exponential, a sine or a cosine, or some other readily integrable function, try U equals the polynomial and dV equals the rest.

(ii) If the integrand involves a logarithm, an inverse trigonometric function, or some other function that is not readily integrable but whose derivative is readily calculated, try that function for U and let d V equal the rest.
(Of course, these "rules" come with no guarantee. They may fail to be helpful if "the rest" is not of a suitable form. There remain many functions that cannot be anti-differentiated by any standard techniques; e.g., e^{x^2} .)
The following two examples illustrate a frequently occurring and very useful phenomenon. It may happen after one or two integrations by parts, with the possible application of some known identity, that the original integral reappears on the right-hand side. Unless its coefficient there is 1, we have an equation that can be solved for that integral.

Example 4.2.9. Evaluate $I = \int \sec^3 x dx$

Solution. We start by integrating by parts: let $U = \sec x, dV = \sec^2 x dx$. Then $dU = \sec x \tan x dx, V = \tan x$. Thus

$$I = \int \sec^3 x dx = \sec x \tan x - \int \sec x \tan^2 x dx = \sec x \tan x - \int \sec x (sec^x - 1) dx$$

$$= \sec x \tan x - \int \sec^3 x dx - \int \sec x dx = \sec x \tan x - I + \ln|\sec x + \tan x|.$$

Solving for I, we have

$$\int \sec^3 x dx = I = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.$$

Example 4.2.10. Evaluate $I = \int e^{ax} \cos bx dx$

Solution. If either $a = 0$ or $b = 0$, the integral is easy to evaluate. So let us assume that $a \neq 0$ and $b \neq 0$. Let $U = e^{ax}, dV = \cos bx dx$. Then $dU = ae^{ax}dx, V = \frac{1}{b}\sin bx$. Thus

$$I = \int e^{ax}\cos bx dx = \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\int e^{ax}\sin bx dx.$$

Now, let $U = e^{ax}, dv = \sin bx dx$. Then $dU = ae^{ax}dx, V = -\frac{\cos bx}{b}$. Thus

$$I = \int e^{ax}\cos bx dx = \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\Big(-\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx dx\Big) = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}I.$$

Thus

$$(1+\frac{a^2}{b^2})I = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx + C_1.$$

Hence

$$\int e^{ax}\cos bx dx = I = \frac{be^{ax}\sin bx + ae^{ax}\cos bx}{a^2+b^2} + C.$$

4.2.3 Integration of Rational Functions

In this subsection we are concerned with integrals of the form

$$\int \frac{P(x)}{Q(x)}dx,$$

where P and Q are polynomials in x. We need normally concern ourselves only with rational functions $\frac{P(x)}{Q(x)}$ where the degree of P is less than that of Q. If the degree of P equals or exceeds the degree of Q, then we can use "`long division`" to express the fraction$\frac{P(x)}{Q(x)}$ as a polynomial plus another fraction $\frac{R(x)}{Q(x)}$, where R, the remainder in the division, has degree less than that of Q. Without loss of generality(WLOG), we will also consider the case where $P(x)$ and $Q(x)$ has no common roots.

Definition 4.2.11. If the degree of the numerator $P(x)$ is less than that of the denominator $Q(x)$, then we say that the rational function $\frac{P(x)}{Q(x)}$ is a **proper function**.

Example 4.2.12. Evaluate $\int \frac{x^3+3x^2}{x^2+1}dx$

Solution. The numerator has degree 3 and the denominator has degree 2, so we use long division to get

$$\frac{x^3+3x^2}{x^2+1} = x+3-\frac{x+3}{x^2+1}.$$

Thus

$$\int \frac{x^3+3x^2}{x^2+1}dx = \int (x+3)dx - \int \frac{x}{x^2+1}dx - \int \frac{3}{x^2+1}dx = \frac{1}{2}x^2+3x-\frac{1}{2}\ln(x^2+1)-3\tan^{-1}x+C.$$

beginexampleEvaluate $\int \frac{x}{2x-1}dx$

Solution. The numerator and the denominator have the same degree 1, so division is again required. We use a method called "`short division`":

$$\frac{x}{2x-1} = \frac{1}{2}\frac{2x}{2x-1} = \frac{1}{2}\frac{2x-1+1}{2X-1} = \frac{1}{2}(1+\frac{1}{2X-1}).$$

Thus

$$\int \frac{x}{2x-1}dx = \frac{1}{2}\int (1+\frac{1}{2x-1})dx = \frac{x}{2}+\frac{1}{4}\ln|2x-1|+C.$$

Linear and Quadratic Denominators

• Suppose that $Q(x)$ is linear. That is $Q(x) = ax+b$. Then

$$\int \frac{c}{ax+b}dx = \frac{c}{a}\ln|ax+b|+C.$$

• Suppose that $Q(x)$ is quadratic,i.e. of degree 2. For purposes of this discussion we can assume that $Q(x)$ is either of the form x^2+a^2 or x^2-a^2, since completing the square and making change of variable can always reduce a quadratic to this form. Since $P(x)$ can be at most a linear function, $P(x) = Ax+B$, we are led to consider the following four integrals

$$\int \frac{xdx}{x^2+a^2}, \quad \int \frac{xdx}{x^2-a^2}, \quad \int \frac{dx}{x^2+a^2}, \quad \int \frac{dx}{x^2-a^2}.$$

The first two integrals yield to the substitution $u = x^2 \pm a^2$. The third is a known integral. In the fourth integral, notice that

$$\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a}.$$

That is $\frac{Ax+Aa+Bx-Ba}{x^2-a^2} = 1$. Equating like coefficients of like terms we have

$$\begin{aligned} A+B &= 0 \\ Aa-Ba &= 1 \end{aligned}$$

Solving, we have $A = \frac{1}{2a}, B = -\frac{1}{2a}$. Thus

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\int \frac{dx}{x-a} - \frac{1}{2a}\int \frac{dx}{x+a} = \frac{1}{2a}\ln|x-a| - \frac{1}{2a}\ln|x+a| + C = \frac{1}{2a}\ln|\frac{x-a}{x+a}| + C.$$

In summary. we have:

$$\begin{aligned} \int \frac{xdx}{x^2+a^2} &= \frac{1}{2}\ln(x^2+a^2)+C \\ \int \frac{xdx}{x^2-a^2} &= \frac{1}{2}\ln(x^2-a^2)+C \\ \int \frac{dx}{x^2+a^2} &= \frac{1}{a}\tan^{-1}x+C \\ \int \frac{dx}{x^2-a^2} &= \frac{1}{2a}\ln|\frac{x-a}{x+a}|+C \end{aligned}$$

The technique used above, involving the writing of a complicated fraction as a sum of simpler fractions, is called the **Method of Partial Fractions**. Suppose that a polynomial $Q(x)$ is of degree n and that its highest degree term is x^n (with coefficient 1). Suppose also that Q factors into a product of n distinct linear (degree 1) factors, say,

$$Q(x) = (x-a_1)(x-a_2)...(x-a_n),$$

where $a_i \neq a_j$ if $i \neq j, 1 \leq i, j \leq n$. If $P(x)$ is a polynomial of degree smaller than n, then $\frac{P(x)}{Q(x)}$ has a **partial fraction decomposition** of the form

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_n}{x-a_n},$$

where $A_1, A_2, \cdots, A_n$ are constants to be determined.

Example 4.2.13. $\int \frac{x+4}{x^2-5x+6}dx$

Solution. Note that

$$\frac{x+4}{x^2-5x+6} = \frac{x+4}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}.$$

That is,

$$x + 4 = Ax + Bx - 3A - 2B.$$

That is $A + B = 1$ and $-3A - 2B = 4$, and upon solving we have $A = -6, B = 7$. Thus

$$\int \frac{x+4}{x^2-5x+6}dx = -6\int \frac{1}{x-2}dx + 7\int \frac{1}{x-3}dx = -6\ln|x-2| + 7\ln|x-3| + C.$$

Example 4.2.14. $\int \frac{x^3+2}{x^3-3}dx$

Solution. The numerator and denominator have same degree, so we must divide:

$$\int \frac{x^3+2}{x^3-3}dx = \int (1 + \frac{x+2}{x^3-x^2})dx = x + \int \frac{x+2}{x^3-x^2}dx.$$

Using the method of partial fractions, we have:

$$\frac{x+2}{x^3-x^2} = \frac{x+2}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}.$$

Thus we have $A + B + C = 0, B - C = 0, -A = 2$, and thus $A = -2, B = \frac{3}{2}$ and $C = \frac{1}{2}$. Thus

$$\int \frac{x^3+2}{x^3-3}dx = x - 2\int \frac{dx}{x} + \frac{3}{2}\int \frac{3}{x-1}dx + \frac{1}{2}\int \frac{3}{x+1}dx = x - 2\ln|x| + \frac{3}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + C.$$

We next consider a rational function whose denominator has a quadratic factor that is a sum of sqaures and cannot be further factored into a product of real linear factors.

Example 4.2.15. Evaluate $\int \frac{2+3x+x^2}{x(x^2+1)}dx$

Solution. No division is required here since degree of numerator is less than degree of denominator. The appropriate form of decomposition turns out to be

$$\frac{2+3x+x^2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}.$$

Equating coefficients, we have $A + B = 1, C = 3, A = 2$. Hence $A = 2, B = -1$, and $C = 3$. Thus

$$\int \frac{2+3x+x^2}{x(x^2+1)}dx = 2\int \frac{1}{x}dx - \int \frac{x}{x^2+1}dx + 3\int \frac{1}{x^2+1}dx = 2\ln|x| - \frac{1}{2}\ln(x^2+1) + 3\tan^{-1}x + C.$$

Completing the Square

Quadratic expressions of the form $Ax^2 + Bx + C$ can be written as

$$\begin{aligned} Ax^2 + Bx + C &= A(x^2 + \tfrac{B}{A}x + \tfrac{C}{A}) \\ &= A(x^2 + \tfrac{B}{A}x + \tfrac{B^2}{4A^2} + \tfrac{C}{A} - \tfrac{B^2}{4A^2}) \\ &= A(x + \tfrac{B}{2A})^2 + \tfrac{4AC-B^2}{4A} \end{aligned}$$

The substitution $u = x + \frac{B}{2A}$ should then be made.

Example 4.2.16. Evaluate$I = \int \frac{1}{x^3+1}dx$

Solution. Note that $Q(x) = x^3 + 1 = (x+1)(x^2 - x + 1)$. The latter factor has no real roots and hence has no real linear sub-factors. We have

$$\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)} = \frac{}{x+1} + \frac{Bx+C}{x^2-x+1}.$$

Equating coefficients we have $A+B = 0, -A+B+C = 0, A+C = 1$. Hence $A = \frac{1}{3}, B = -\frac{1}{3}, C = \frac{2}{3}$. We thus have

$$I = \int \frac{1}{x^3+1}dx = \frac{1}{3}\int \frac{dx}{x+1} - \frac{1}{3}\int \frac{dx}{x^2-x+1} = \frac{1}{3}\ln|x+1| - \frac{1}{3}\int \frac{dx}{x^2-x+1}.$$

Completing squares, $x^2 - x + 1 = (x - \frac{1}{2})^2 + \frac{3}{4}$. Thus

$$I = \int \frac{1}{x^3+1}dx = \frac{1}{3}\ln|x+1| - \frac{1}{3}\int \frac{x - \frac{1}{2} - \frac{3}{2}}{(x-\frac{1}{2})^2 + \frac{3}{4}}.$$

Now letting $u = x - \frac{1}{2}$, we have $du = dx$. Thus

$$I = \int \frac{1}{x^3+1}dx = \frac{1}{3}\ln|x+1| - \frac{1}{3}\int \frac{u}{u^2+\frac{3}{4}} + \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}$$

$$= ln(u^2 + \frac{3}{4}) + \frac{1}{2}\frac{2}{\sqrt{3}}\tan^{-1}(\frac{2u}{\sqrt{3}}) + C = \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln(x^2-x+1) + \frac{1}{\sqrt{3}}\tan^{-1}(\frac{2x-1}{\sqrt{3}}) + C.$$

Denominators with Repeated Factors

We require one final refinement of the method of partial fractions. If any of the linear or quadratic factors of $Q(x)$ is repeated (say, m times), then the partial fraction decomposition of $\frac{P(x)}{Q(x)}$ requires m distinct fractions corresponding to that factor. The denominators of these fractions have exponents increasing from 1 to m, and the numerators are all constants where the repeated factor is linear or linear where the repeated factor is quadratic.

Example 4.2.17. Evaluate$I = \int \frac{1}{x(x-1)^2}dx$

Solution. The appropriate partial fraction decomposition is

$$\frac{1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$

Equating coefficients of x^2, x, and 1, we have

$$\begin{aligned} A + B &= 0 \\ -2A + B + C &= 0 \\ A &= 1 \end{aligned}$$

Hence $A = 1, B = -1, C = 1$, and

$$I = \int \frac{1}{x(x-1)^2}dx = \frac{1}{x}dx - \frac{1}{x-1}dx + \frac{1}{(x-1)^2}dx = \ln|x| - \ln|x-1| - \frac{1}{x-1} + C$$

$$= \ln|\frac{x}{x-1}| - \frac{1}{x-1} + C.$$

4.2.4 Inverse Substitutions

The Inverse Trigonometric Substitutions

Three very useful inverse substitutions are:

$$x = a\sin\theta, x = a\tan\theta, x = a\sec\theta.$$

These corresponds to the direct substitutions:

$$\theta = \sin^{-1}\frac{x}{a}, \theta = \tan^{-1}\frac{x}{a}, \theta = \sec^{-1}\frac{x}{a} = \cos^{-1}\frac{a}{x}.$$

Inverse Sine Substitutions
Integrals involving $\sqrt{a^2 - x^2}$ where $a > 0$ can be reduced to a simpler form by means of the substitution $x = a\sin\theta$. Observe that $\sqrt{a^2 - x^2}$ makes sense if $-a \leq x \leq a$, which corresponds to $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Since $\cos\theta \geq 0$, for such θ we have

$$\sqrt{a^2 - x^2} = \sqrt{a^2(1 - \sin^2\theta)} = \sqrt{a^2\cos^2\theta} = a\cos\theta.$$

Therefore $\cos\theta = \frac{\sqrt{a^2-x^2}}{a}$ and $\tan\theta = \frac{x}{\sqrt{a^2-x^2}}$.

Example 4.2.18. Evaluate $\int \frac{1}{(5-x^2)^{\frac{3}{2}}}dx$

Solution. Let $x = \sqrt{5}\sin\theta$. Then $dx = \sqrt{5}\cos\theta d\theta$. Thus

$$4\int \frac{1}{(5-x^2)^{\frac{3}{2}}}dx = \int \frac{\sqrt{5}\cos\theta d\theta}{5^{\frac{3}{2}}\cos^3\theta} = \frac{1}{5}\int \sec^2\theta d\theta = \frac{1}{5}\tan\theta + C = \frac{1}{5}\frac{x}{\sqrt{5-x^2}} + C.$$

Inverse Tangent Substitutions
Integrals involving $\sqrt{a^2 + x^2}$ or $\frac{1}{x^2+a^2}$ $(a > 0)$ are often simplified by the substitution $x = a\tan\theta$. Since x can take any any real value, we have $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ or $\sec\theta > 0$ and

$$\sqrt{a^2 + x^2} = a\sqrt{1 + \tan^2\theta} = a\sec\theta.$$

Therefore $\sin\theta = \frac{x}{\sqrt{a^2+x^2}}$ and $\cos\theta = \frac{a}{\sqrt{a^2+x^2}}$.

Example 4.2.19. Evaluate $\int \frac{1}{\sqrt{4+x^2}}dx$

Solution. Let $x = 2\tan\theta$. Then $dx = 2\sec^2\theta d\theta$. Thus

$$\int \frac{1}{\sqrt{4+x^2}}dx = \int \frac{2\sec^2\theta}{2\sec\theta}d\theta = \int \sec\theta\ d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$= \ln|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}| + C = \ln(\sqrt{4+x^2} + x) + C_1,$$

where $C_1 = C - \ln 2$.

Example 4.2.20. Evaluate $\int \frac{1}{(1+9x^2)}dx$

Solution. Let $3x = \tan\theta$. Then $3dx = \sec^2\theta d\theta$ and $1+9x^2 = \sec^\theta$. Thus

$$\int \frac{1}{(1+9x^2)}dx = \frac{1}{3}\int \frac{\sec^2\theta d\theta}{\sec^4\theta} = \frac{1}{3}\int \cos^2\theta d\theta = \frac{1}{6}(\theta + \sin\theta\cos\theta) + C$$

$$= \frac{1}{6}\tan^{-1}3x + \frac{1}{6}\frac{3x}{\sqrt{1+9x^2}}.\frac{1}{\sqrt{1+9x^2}} + C$$

$$= \frac{1}{6}\tan^{-1}3x + \frac{1}{2}.\frac{x}{1+9x^2} + C.$$

Inverse Secant Substitutions
Integrals involving $\sqrt{x^2-a^2}$ (where $a > 0$) can be simplified by using the substitution $x = a\sec\theta$. But we must be careful with this substitution. Although $\sqrt{x^2-a^2} = a\sqrt{\sec^2\theta - 1} = a\sqrt{\tan^2\theta} = a|\tan\theta|$, we cannot always drop the absolute value from the tangent. Observe that $\sqrt{x^2-a^2}$ makes sense for $x \geq a$ and for $x \leq -a$.

- If $x \geq a$, then $0 \leq \theta = \sec^{-1}\frac{x}{a} = \cos^a x < \frac{\pi}{2}$, and $\tan\theta \geq 0$.
- If $x \leq -a$, then $\frac{\pi}{2} < \theta = \sec^{-1}\frac{x}{a} = \cos^a x \leq \pi$, and $\tan\theta \leq 0$.

In the first case $\sqrt{x^2-a^2} = a\tan\theta$ and in the second case $\sqrt{x^2-a^2} = -a\tan\theta$.

Example 4.2.21. Find $I = \int \frac{dx}{\sqrt{x^2-a^2}}$

Solution. Assume $x \geq a$.
If $x = \sec\theta$, then $dx = a\sec\theta\tan\theta d\theta$ and $\sqrt{x^2-a^2} = a\tan\theta$. Thus,

$$I = \int \frac{dx}{\sqrt{x^2-a^2}} = \int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C = \ln|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}| + C = \ln|x + \sqrt{x^2-a^2}| + C_1,$$

where $C_1 = C - \ln a$.
If $x \leq -a$, let $u = -x$ so that $u \geq a$ and $du = -dx$. We have

$$I = -\int \frac{du}{\sqrt{u^2-a^2}} = -\ln|u + \sqrt{u^2-a^2}| + C_1 = \ln|\frac{1}{-x+\sqrt{x^2-a^2}}.\frac{x+\sqrt{x^2-a^2}}{x+\sqrt{x^2-a^2}}| + C_1$$

$$= \ln|\frac{x+\sqrt{x^2-a^2}}{-a^2}| + C_1 = \ln|x + \sqrt{x^2-a^2}| + C_2,$$

where $C_2 = C_1 - 2\ln a$. Thus, in either case, we have

$$I = \ln|x + \sqrt{x^2-a^2}| + C.$$

Example 4.2.22. Find $I = \int \frac{1}{\sqrt{2x-x^2}}dx$

Solution.
$$I = \int \frac{1}{\sqrt{2x - x^2}} dx = \int \frac{dx}{\sqrt{1 - (1 - 2x + x^2)}} = \int \frac{dx}{\sqrt{1 - (x-1)^2}}.$$
Now, if we let $u = x - 1$, then $du = dx$. Thus
$$I = \int \frac{1}{\sqrt{2x - x^2}} dx = \int \frac{dx}{\sqrt{1 - (x-1)^2}} = \int \frac{du}{\sqrt{1 - u^2}} = \sin^{-1} u + C = \sin^{-1}(x-1) + C.$$

beginexampleFind $I = \int \frac{x}{4x^2 + 12x + 13} dx$

Solution.
$$I = \int \frac{x}{4x^2 + 12x + 13} dx = \int \frac{xdx}{4(x^2 + 3x + \frac{9}{4} + 1} = \frac{1}{4} \int \frac{xdx}{(x + \frac{3}{2})^2 + 1}.$$
Let $u = x + \frac{3}{2}$. Then $du = dx$. Thus
$$I = \int \frac{x}{4x^2 + 12x + 13} dx = \int \frac{xdx}{4(x^2 + 3x + \frac{9}{4} + 1)} = \frac{1}{4} \int \frac{xdx}{(x + \frac{3}{2})^2 + 1}$$
$$= \frac{1}{4} \int \frac{udu}{u^2 + 1} - \frac{3}{8} \int \frac{du}{u^2 + 1}.$$
In the first integral, let $v = u^2 + 1$, thus $dv = 2udu$. Therefore,
$$I = \int \frac{x}{4x^2 + 12x + 13} dx = \frac{1}{4} \int \frac{udu}{u^2 + 1} - \frac{3}{8} \int \frac{du}{u^2 + 1} = \frac{1}{8} \int \frac{dv}{v} - \frac{3}{8} \tan^{-1} u$$
$$= \frac{1}{8} \ln |v| - \frac{3}{8} \tan^{-1} u + C = \frac{1}{8} \ln(4x^2 + 12x + 13) - \frac{3}{8} \tan^{-1}(x + \frac{3}{2}) + C_1,$$
where $C_1 = C - \frac{\ln 4}{8}$.

4.2.5 Inverse Hyperbolic Substitutions

As an alternative to the inverse secant substitution $x = a \sec \theta$ to simplify integrals involving $\sqrt{x^2 - a^2}$ (where $x \geq a > 0$) we can use the inverse hyperbolic cosine substitution $x = a \cosh u$. Since $\cosh^2 u - 1 = \sinh^2 u$, this substitution produces $\sqrt{x^2 - a^2} = a \sinh u$. To express u in terms of x, we need the result, note that
$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}), \quad x \geq 1.$$

Example 4.2.23. Find $I = \int \frac{dx}{\sqrt{x^2 - a^2}}$, (where $a > 0$)

Solution. Note that this problem has been solved using the inverse secant substitution. Again we assume $x \geq a$(the case where $x \leq -1$ can be handled similarly.) Using the substitution $x = a \cosh u$, so that $dx = a \sinh u du$, we have
$$I = \int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{a \sinh u}{a \sinh u} du = \int du = u + C = \cosh^{-1} \frac{x}{a} + C = \ln(\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}) + C$$

$$= \ln(x + \sqrt{x^2 - a^2}) + C_1,$$

where $C_1 = C - \ln a$.

Similarly, the inverse hyperbolic substitution $x = a \sinh u$ can be used instead of the inverse tangent substitution $x = a \tan\theta$ to simplify integrals involving $\sqrt{x^2 + a^2}$ or $\frac{1}{x^2+a^2}$. In this case, we have $dx = a \cosh u du$ and $x^2 + a^2 = a^2 \cosh^2 u$, and we may need the result that

$$\sinh^2 x = \ln(x + \sqrt{x^2 + 1}),$$

which is valid for all x.

4.2.6 Other Inverse Substitutions

Integrals involving $\sqrt{ax + b}$ can be made simpler with the substitution $ax + b = u^2$.

Example 4.2.24. Find $\int \frac{1}{1+\sqrt{2x}} dx$

Solution. Let $2x = u^2$. Then $2dx = 2udu$. Thus

$$\int \frac{1}{1+\sqrt{2x}} dx = \int \frac{u}{1+u} du = \int \frac{1+u-1}{1+u} du = \int (1 - \frac{1}{1+u}) du.$$

Let $v = 1 + u$ and hence $dv = du$. Thus

$$\int \frac{1}{1+\sqrt{2x}} dx = \int (1 - \frac{1}{1+u}) du = u - \int \frac{dv}{v} = u - \ln|v| + C = \sqrt{2x} - \ln(1 + \sqrt{2x}) + C.$$

Sometimes integrals involving $\sqrt[n]{ax + b}$ will be much simplified by the hybrid substitution $ax+b = u^n, adx = nu^{n-1}du$.

Example 4.2.25. Find $\int \frac{x}{1+\sqrt[3]{3x+2}} dx$

Solution. Let $3x + 2 = u^3$. Then $3dx = 3u^2du$. Thus

$$\int \frac{x}{1+\sqrt[3]{3x+2}} dx = \int \frac{u^3 - 2}{3u} u^2 du = \frac{1}{3} \int (u^4 - 2u) du = \frac{1}{3} (\frac{u^5}{5} - u^2) + C$$

$$= \frac{1}{3} \Big(\frac{(\sqrt[3]{3x+2})^5}{5} - (\sqrt[3]{3x+2})^2 \Big) + C.$$

If more than one fractional power is present, it may be possible to eliminate all of them at once.

Example 4.2.26. Evaluate $\int \frac{1}{x^{\frac{1}{2}}(1+x)^{\frac{1}{3}}} dx$

Solution. Eliminate both the square root and the cube root by using the inverse substitution $x = u^6$ (6 is chosen because it is the LCM of 2 and 3). Let $x = u^6$. Then $dx = 6u^5du$. Thus

$$\int \frac{1}{x^{\frac{1}{2}}(1+x)^{\frac{1}{3}}} dx = 6 \int \frac{u^5 du}{u^3(1+u^2)} = 6 \int \frac{u^2}{1+u^2} du$$

$$= 6\int(1-\frac{1}{1+u^2})du = 6(u-\tan^{-1}u)+C = 6(x^{\frac{1}{6}}-\tan^{-1}x^{\frac{1}{6}})+C.$$

The $tan(\frac{\theta}{2})$ Substitution
There is a certain special substitution that can transform an integral whose integrand is a rational function of $sin\theta$ and $cos\theta$ (i.e., a quotient of polynomials in $sin\theta$ and $cos\theta$ into a rational function of x. The substitution is $x=\tan\frac{\theta}{2}$ or, equivalently, $\theta = 2\tan^{-1}x$. Observe that

$$\cos^2\frac{\theta}{2} = \frac{1}{\sec^2}\frac{\theta}{2} = \frac{1}{1+\tan^2\frac{\theta}{2}} = \frac{1}{1+x^2},$$

so

$$\cos\theta = 2\cos^2\frac{\theta}{2}-1 = \frac{2}{1+x^2}-1 = \frac{1-x^2}{1+x^2}$$

$$\sin\theta = 2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2} = 2\tan\frac{\theta}{2}\cos^2\frac{\theta}{2} = \frac{2x}{1+x^2}$$

Also $dx = \frac{1}{2}\sec^2\frac{\theta}{2}d\theta$, so

$$d\theta = 2\cos^2\frac{\theta}{2}dx = \frac{2dx}{1+x^2}.$$

Example 4.2.27. Evaluate $\int\frac{1}{2+\cos\theta d\theta}$

Solution. Let $x=\tan\frac{\theta}{2}$. Then $\cos\theta = \frac{1-x^2}{1+x^2}$ and $d\theta = \frac{2dx}{1+x^2}$. Thus

$$\int\frac{1}{2+\cos\theta}dx = \int\frac{\frac{2dx}{1+x^2}}{2+\frac{1-x^2}{1+x^2}} = 2\int\frac{1}{3+x^3}$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}+C = \frac{2}{\sqrt{3}}\tan^{-1}(\frac{1}{3}\tan\frac{\theta}{2})+C.$$

Example 4.2.28. Evaluate $\int\frac{dx}{\cos x}dx$

Solution. Taking into account that $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2dt}{1+t^2}$, we find that

$$\int\frac{dx}{\cos x}dx = \int\frac{\frac{2dt}{1+t^2}}{2+\frac{1-t^2}{1+t^2}} = 2\int\frac{dt}{1+t^2} = \ln|\frac{1+t}{1-t}|+C = \ln|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}|+C.$$

4.3 The Definite Integral

The definite integral of $f(x)$ over $[a,b]$, denoted by $\int_a^b f(x)dx$ is a `number`; it is not a function of x. It depends on the numbers a and b and on the particular function f, but not on the variable x. It is defined

$$\int_a^b f(x)dx = [F(x)+C]|_a^b = F(b)-F(a), \qquad (\star)$$

where F is an antiderivative of f.

Definition 4.3.1. Let $f(x)$ be a function whose domain contains the closed interval $[a, b]$. Suppose $F(x)$ is an antiderivative for $f(x)$. Then the **definite integral** of $f(x)$ from $x = a$ to $x = b$, denoted by $\int_a^n f(x)dx$ is defined as

$$\int_a^b f(x)dx = [F(x) + C]|_{x=a}^{x=b} = F(b) - F(a).$$

Here, a is called the **lower bound/limit of integration** and b is called the **upper bound/limit of integration**.

Observe that the difference $F(b) - F(a)$ does not depend on C, because

$$(F(b) + C) - (F(a) + C) = F(b) - F(a).$$

Formula $(\star)$ is called the **Newton-Leibniz formula**. Note that in $(\star)$ if $a = b$, then the integral is zero. That is $\int_a^a f(x)dx = 0$.
The techniques of integration can be used to evaluate definite integrals.

4.3.1 Properties of the Definite Integral

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
2. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$
3. $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
4. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \quad$ where $\quad c \in [a, b]$

If a function F is an antiderivative of f on the half-interval $[a, \infty)$ and $\lim_{x\to\infty} F(x)$ exists, then

$$\int_a^\infty f(x)dx = \lim_{x\to\infty} F(x) - F(a) = \lim_{x\to\infty} \int_a^x f(t)dt.$$

Similarly

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx.$$

Such integrals are called **improper integrals**.

Example 4.3.2. Find $\int_0^1 \frac{dx}{\sqrt{1-x}}$

Example 4.3.3. Solution. The integrand is not defined at $x = 1$, and so as an improper integral, we can write

$$\int_0^1 \frac{dx}{\sqrt{1-x}} = \lim_{x\to 1^-} \int_0^x \frac{dt}{\sqrt{1-t}} = \lim_{x\to 1^-} (-2\sqrt{1-x}) - (-2) = 2.$$

Example 4.3.4. Find $\int_1^\infty \frac{dx}{x^2}$

Example 4.3.5. Solution. The interval of integration is the infinite interval $[0, \infty)$.

$$\int_1^\infty \frac{dx}{x^2} = \lim_{x\to\infty} \int_1^\infty \frac{dt}{t^2} = \lim_{x\to\infty} (-\frac{1}{x}) - (-1) = 1.$$

Example 4.3.6. Evaluate $\int_0^1 2x(x^2 + 1)^5 dx$

Example 4.3.7. Solution:Method 1. Let $u = x^2 + 1$, $du = 2xdx$. Then

$$\int_0^1 2x(x^2+1)^5 dx = \int u^5 du = \frac{u^6}{6} + C = \frac{x^2+1}{6} + C.$$

Consequently,

$$\int_0^1 2x(x^2+1)^5 dx = \frac{x^2+1}{6}|_0^1 = \frac{2^6}{6} - \frac{1^6}{6} = \frac{21}{2}.$$

Method 2. Let $u = x^2 + 1$, $du = 2xdx$; however, we also apply the substitution to the limits of integration as well. When $x = 0$, we have $u = 0^2+1 = 1$; and when $x = 1$, we have $u = 1^2+1 = 2$. Therefore

$$\int_0^1 2x(x^2+1)^5 dx = \int_1^2 u^5 du = \frac{u^6}{6}|_1^2 = \frac{2^6}{6} - \frac{1^6}{6} = \frac{21}{2}.$$

4.4 Applications of the Definite Integral

We give concepts of length, area and volume analytic definitions using the concept of the definite integral.

4.4.1 Area and the Definite Integral

Area as a Limit

Let f be a continuous, positive valued function on $[a, b]$ and suppose we want to compute the area of f under the graph of f from $x = a$ to $x = b$. We may approximate the area by partitioning the region into n trapezia (subdivide $[a, b]$ into n equal intervals each of length Δx) and finding the area of each trapezia and add the areas up.

$$A = \lim_{n\to\infty}(\sum_{i=1}^{n} f(x_i)\Delta x),$$

where $\Delta x = \frac{b-a}{n}$ and x_i is the right-hand endpoint of the ith interval $[x_{i-1}, x_i]$.
Note that the definite integral exists if the limit is a finite number. The actual area A is given by the definite integral

$$A = \int_a^b f(x)dx. \qquad (\star).$$

Note that the approximation to area approaches the actual area as $n \to \infty$. Thus

$$A = \int_a^b f(x)dx = \lim_{n\to\infty}(\sum_{i=1}^{n} f(x_i)\Delta x).$$

It can be shown that the partitioning points x_i need not be equally spaced as long as the longest interval nears zero as n increases without bound. The definite integral defined here is called a **Riemann integral**.
If $f(x) < 0$ on $[a, b]$, then the integral $(\star)$ is a non-positive number,$-A$. Since the area is nonnegative, we define

$$A = \int_a^b |f(x)|dx.$$

Let f_1, f_2 be continuous functions with $f_1(x) \geq f(x_2)$ for $x \in [a, b]$. Then the area A of the region bounded by the curves $y = f_1(x)$ and $y = f_2(x)$ and the vertical lines $x = a, x = b$ is

$$A = \int_a^b [f(x) - g(x)]dx \qquad (\star\star)$$

More generally, consider a continuous function f with a graph that crosses the x-axis at finitely many points $x_i, \quad i = 1, 2, ..., k$ between a and b. Then we write

$$A = \int_a^b f(x)dx = \int_a^{x_1} f(x)dx + \int_{x_1} a^{x_2} f(x)dx + \cdots + \int_{x_k}^b f(x)dx.$$

Hence, we see that $\int_a^b f(x)dx$ is equal to the area under $y = f(x)$ above the x-axis, minus the area over $y = f(x)$ below the x-axis.

Finally, let f_1, f_2 be continuous, with $f_1(y) \geq f_2(y)$ for $y \in [c, d]$. Then the area A of the region bounded by the curves $x = f_1(y)$ and $x = f_2(y)$ and the vertical lines $y = c$ and $y = d$ is

$$A = \int_c^d [f_1(y) - f_2(y)]dy.$$

Example 4.4.1. Find the area A of the region bounded by the curves $y = x^\alpha$ and $x = y^\alpha$.

Solution. The points of intersection of these curves are $(0, 0)$ and $(1, 1)$. Because $x^{\frac{1}{\alpha}} \geq x^\alpha$ for all $x$in $[0, 1]$, it follows that

$$A = \int_0^1 (x^{\frac{1}{\alpha}} - x^\alpha)dx = \left(\frac{\alpha x^{\frac{\alpha+1}{\alpha}}}{\alpha + 1} - \frac{x^{\alpha+1}}{\alpha + 1}\right) = \frac{\alpha - 1}{\alpha + 1}.$$

Example 4.4.2. Find the area A of the region bounded by the curve $y = \sin x$, $x \in [0, 2\pi]$ and the x-axis.

Solution. $A = \int_0^{2\pi} |\sin x|dx = \int_0^\pi \sin x dx + |\int_\pi^{2\pi} \sin x dx| = -(\cos\pi - \cos 0) + |-(\cos 2\pi - \cos\pi)| = 2 + |-2| = 4.$

Example 4.4.3. Find the area A of the region bounded by the parabola $f(x) = 2 - x^2$ and the line $g(x) = -x$.

Solution. Notice that the curves intersect at the points $(-1, 1)$ and $(2, -2)$ by solving $2-x^2 = -x$.

$$A = \int_{-1}^{2} [f(x) - g(x)]dx = \int_{-1}^{2} (2 - x^2 + x)dx = \cdots = \frac{9}{2}.$$

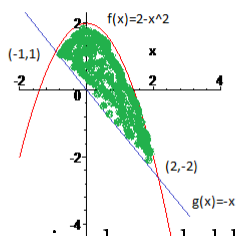

Figure 4.1: Area of region bounded by f and g

Example 4.4.4. Find the area A of the region bounded by the x-axis and $f(x) = x^2 + x - 6$.

Solution. The zeros of $f(x) = x^2 + x - 6$ are $(-3, 0)$ and $(2, 0)$. Since the x axis has equation $y = 0$, the desired area can be considered as the area of a region between the two curves $y = 0$ and $f(x) = x^2 + x - 6$. Thus

$$A = \int_{3}^{2} [0 - (x^2 + x - 6)]dx = \frac{125}{6}.$$

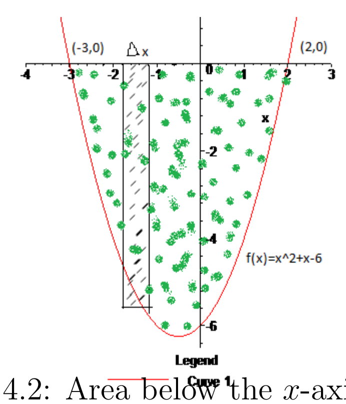

Figure 4.2: Area below the x-axis

Example 4.4.5. Find the area A of the region bounded by the x-axis, $f(x) = x^3, a = -3$ and $b = 3$.

Solution.

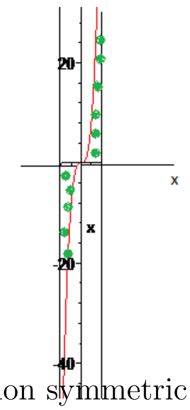

Figure 4.3: Area of region symmetric to the x-axis

From the graph, part of the region falls below the x-axis and part above. Note that

$$\int_{-3}^{3} x^3 dx = 0.$$

This is incorrect. Since the region below the x-axis gives a negative value for the area and the part above the x-axis gives a positive value for the area, then the area A is

$$A = |\int_{-3}^{0} x^3 dx| + \int_{0}^{3} x^3 dx = |-\frac{81}{4}| + \frac{81}{4} = \frac{81}{2}.$$

Another way of approaching the problem is to realize that the region below the x-axis is symmetric to the region above the x-axis and therefore

$$A = 2\int_{0}^{3} x^3 dx \quad or \quad A = 2|\int_{-3}^{0} x^3 dx|.$$

Summary. The area of a region bounded by two or more curves can be found by integration. The integrand is $g(x) - f(x)$ when $g(x) \geq f(x)$ for all x in the interval of integration. The bounds of integration are usually found by determining the points of intersection of the curves. In some cases it may be necessary to use more than one integral. If $f(x)$ or $(g(x) - f(x))$ crosses the x-axis in $[a, b]$, then more than one integral may be necessary.

4.5 Exercises and Some Solved Problems

1. Evaluate

(i). $\int \cos^2 x dx$ **Soln**: $\int \cos^2 x dx = \frac{1}{2}\int(1 + \cos 2x)dx = \frac{x}{2} + \frac{1}{4}\sin 2x + c.$

(ii). $\int \sin^4 x dx$

Soln: $\int \sin^4 x dx = \frac{1}{4}\int (1-\cos 2x)^2 dx = \frac{1}{4}\int (1-2\cos 2x+\cos^2 2x)dx = \frac{x}{4} - \frac{1}{4}\sin 2x + \frac{1}{8}\int (1+\cos 4x)dx = \frac{x}{4} - \frac{1}{4}\sin 2x + \frac{x}{8} + \frac{1}{32}\sin 4x + c = \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c.$

(iii). $^2 \cos(2x^3+1)dx$

(iv). $\int \frac{\ln x}{x} dx$

(v). $\int x \cos x dx$

(vi). $\int \frac{\ln(\ln x)}{x} dx$

(vii). $\int \frac{d\theta}{\cos\theta(1+\sin\theta)}$

(viii). $\int \sqrt{a^2-x^2}\ dx$

2. Find the area of the plane region bounded by the given curves.

(i). $y = x, y = x^2$

(ii). $y = x^3, y = x$

(iii). $y = \frac{1}{x}, 2x + 2y = 5$

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